NOIR PREMIERE. **CHEAT SHEET.** Every formula. Its prologue. Its confluence.

How to read a row. CIRCUIT is the command term. **FORMULA** is the rule. **HOLDS WHEN** is the prologue: the conditions for the rule to be true, and what does not matter. **SUMMON WHEN** is the confluence: the cue in the question that fires the circuit, the reflex, and the past papers it appeared in (paper, question). a is always the leading coefficient; $\star$, $\triangle$, d are always gaps between x-coordinates of intersections, never slanted distances; D is a discriminant. Blurt the prologue before the formula, every time.

Colour key

CHAPTER — Card edge, circuit name and curve colour follow the chapter (map on the right).

HOLDS WHEN — The prologue. Slate. Read it first.

SUMMON WHEN — The confluence. Deep red.

Trigger. the cue in the question. **Reflex.** the move it fires. *Seen:* paper, question.

$S = \frac{|a|}{6}\star^3$ headline result of the row.

Diagrams: chapter-colour curve, **second colour** for the second region or curve, **red dot** for the key point.

Chapter map

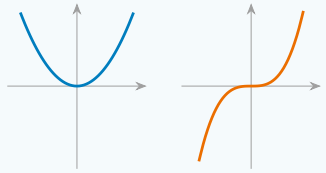
I — 14 circuits — Polynomials (graphs)	II — 5 circuits — Polynomials (algebra)	III — 4 circuits — Sequences and series	IV — 10 circuits — Trigonometry
V — 5 circuits — Geometry (circles)	VI — 3 circuits — Logic	VII — 3 circuits — Function transformation	+ — 1 circuits — After “math”

Area ladder: the lobe formulas at a glance

SHAPE	SETUP	AREA	ROW
Quadratic meets any line	$f - l$ quadratic, $\star = \sqrt{D}/\lvert a\rvert$	$\frac{\lvert a\rvert}{6}\star^3$	$\star^3$ chord area (F1, F2)
Cubic, tangent line meets it once more	$f - l = a(x-m)^2(x-n)$, $\star = \lvert n-m\rvert$	$\frac{\lvert a\rvert}{12}\star^4$	$\star^4/12$ (F5)
Cubic, line through the inflection point	three equal gaps d, each lobe	$\frac{\lvert a\rvert}{4}d^4$	$d^4/4$ (F6)
Cubic, line at three points, any spacing	gaps $\star$, $\triangle$	$\frac{\lvert a\rvert}{6}\star^3\left(\frac{\star}{2}+\triangle\right)$	unequal lobes (F7)
Quartic W, middle hump	$a(x-\alpha)^2(x-\beta)^2$	$\frac{\lvert a\rvert}{30}\star^5$	quartic lobes (F8 to F11)
Quartic, one triple root	$a(x-\alpha)^3(x-\beta)$	$\frac{\lvert a\rvert}{20}\star^5$	quartic lobes (F8 to F11)
Box from a stationary corner on $y = ax^n$	rectangle to (k, ak^n)	under the curve $= \frac{1}{n+1}$ of the box	$n:1$ box

I. POLYNOMIALS (GRAPHS) — 14 circuits

end behaviour



FORMULA

Even degree: $\lim_{x\to\infty} f \cdot \lim_{x\to-\infty} f > 0$ (same sign both ends). Odd degree: product < 0, so at least one real root. Quadratic: one shape. Cubic: three shapes per sign of a, sorted by the sign of the gradient at the inflection point (worm, flat, climb). Turning points alternate max, min: counts differ by at most 1; a quintic has 0, 2 or 4.

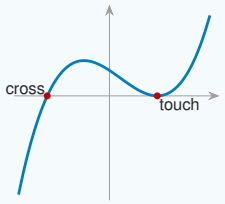
HOLDS WHEN

Any polynomial, $a \neq 0$. Quadratics with the same a always overlap by translation and reflection.

SUMMON WHEN

Trigger. degree parity; “positive for all x”; “which sketch is impossible”. **Reflex.** $f - g > 0$ everywhere forces even degree, so the odd leading terms must cancel.
Seen: 2020 P2 Q18; 2021 P2 Q9; Spec P2 Q14

touch or cross



FORMULA

Root of multiplicity m: m even touches and turns ($\cup$ or $\cap$); m odd crosses (flattens through like an S when $m \geq 3$). Repeated root $\Rightarrow f'(a) = 0$. Quartics: $a(x-m)^2(x-n)^2$ symmetric W peaking at the midpoint; $a(x-m)^3(x-n)$ flattens through m; $a(x-m)^4$ flat cup.

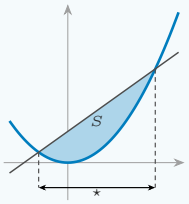
HOLDS WHEN

Read the parity off the factor $(x-a)^m$, never off the degree: $(x-1)^2(x-2)$ is cubic yet touches at 1. The "x-axis" may be any horizontal line.

SUMMON WHEN

Trigger. factorised form; "touches the x-axis"; sketch matching. *Seen:* 2022 P1 Q18

$\star^3$ chord area (F1, F2)



FORMULA

$S = \frac{|a|}{6}\star^3$, $\star = \frac{\sqrt{D}}{|a|}$ with D the discriminant of $f - l$. Memorise also $\int_{-m}^{m}(x^2-m^2)\,dx = -\frac{4}{3}m^3$.

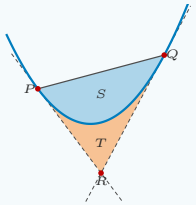
HOLDS WHEN

Quadratic f meeting any line l, horizontal or diagonal: $f - l$ is a quadratic with the same a. The line's gradient, the level k and the roots never enter. Sideways parabolas $y = k\sqrt{x}$ too.

SUMMON WHEN

Trigger. "area enclosed between the curve and the line"; "the graphs do not meet" ($D < 0$: no gap, no area). **Reflex.** read a, find $\star$, cube, over 6. Modulus in the curve: full lobe minus the strip, then symmetry. *Seen:* 2022 P1 Q7; 2019 P1 Q9; 2023 P1 Q2

$S : T = 2 : 1$ (F3)



FORMULA

$S = 2T$: chord region $S = 2\times$ two-tangent region T. Tangents at P, Q meet at R: $x_R = \frac{1}{2}(x_P + x_Q)$; $f'(x_R) =$ gradient of chord $PQ =$ mean of the two tangent gradients; $f'(x_P), f'(x_R), f'(x_Q)$ form an AP.

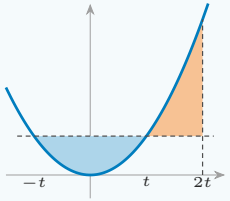
HOLDS WHEN

Parabola and two of its tangents.

SUMMON WHEN

Trigger. "meets the curve at exactly one point" said twice; area between a parabola and two tangents. **Reflex.** intersect the tangents for x_R, average the gradients for $f'(x_R)$. *Seen:* 2021 P1 Q8

1:2:1 and $\sqrt{3}$ (F4)



FORMULA

Horizontal line cuts a parabola, half-width t, vertex at the midpoint. Outer vertical line at $2t$ from the vertex (axis cut $1:2:1$): outer strip $=$ whole lobe. Outer vertical at $\sqrt{3}\,t$ from the vertex: outer strip $=$ half the lobe.

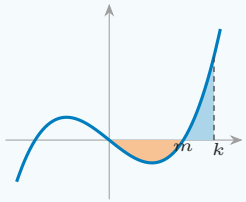
HOLDS WHEN

Any quadratic; ratios survive enlargement. Measure from the vertex, not from the root.

SUMMON WHEN

Trigger. "area R equals area S"; same shading, same area; one region outside the lobe. **Reflex.** lobe width is twice the outer strip width. *Seen:* 2020 P1 Q11

integral zero



FORMULA

Equal areas on opposite sides of a line: $\int_{\text{whole span}} (f-l)\,dx = 0$. For $f = ax(x-m)(x+m)$, lobe $0 \to m$ equal to $m \to k$ forces $k = \sqrt{2}\,m$.

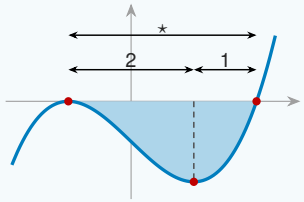
HOLDS WHEN

Regions on opposite sides of the line sharing a boundary; one signed integral replaces two equated ones.

SUMMON WHEN

Trigger. "area R = area S" with the curve crossing the line between them. *Seen:* 2020 P1 Q11; 2018 P1 Q12

$\star^4/12$ (F5)



FORMULA

$S = \dfrac{|a|}{12}\star^4$, $\star$ = gap between the tangent point and the second crossing.

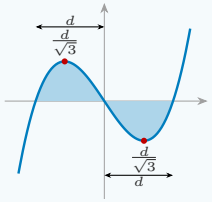
HOLDS WHEN

Cubic, line tangent (repeated root) and meeting once more: $f - l = a(x-m)^2(x-n)$. Any tangent line works; horizontal at a turning point is the case TMUA draws. Worm shape.

SUMMON WHEN

Trigger. cubic tangent to a horizontal line; "touches at its maximum and crosses again". **Reflex.** get $\star$ from the five key points $(2:1)$, fourth power, over 12. *Seen:* NOIR questions; 2021 P1 Q2 setup

$d^4/4$ (F6)



FORMULA

Each of the two lobes: $S = \dfrac{|a|}{4}d^4$, d = common gap of the three intersection x-values.

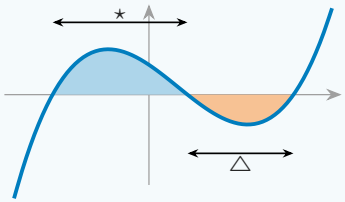
HOLDS WHEN

Line through the inflection point (three equidistant intersections), horizontal or diagonal; a is the leading coefficient, not a first term. Extrema sit at $\pm d/\sqrt{3}$ from the inflection point, not $\pm d/2$. A line through the inflection point meets the cubic 1 or 3 times, never 2.

SUMMON WHEN

Trigger. "$y = mx$ and $y = x^3$"; any line through the centre of a cubic; three equally spaced roots. **Reflex.** two lobes, each $\frac{|a|}{4}d^4$. *Seen:* 2020 P1 Q14

unequal lobes (F7)



FORMULA

Gaps $\star$, $\triangle$: $S_\star = \dfrac{|a|}{6}\star^3\left(\frac{\star}{2}+\triangle\right)$, $S_\triangle = \dfrac{|a|}{6}\triangle^3\left(\frac{\triangle}{2}+\star\right)$. Collapses to F6 when $\star = \triangle$.

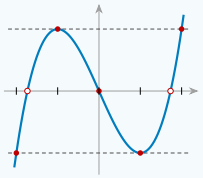
HOLDS WHEN

Cubic meeting a line at three points, any spacing; the bracket is the distance from the midpoint of the lobe's own base to the far root.

SUMMON WHEN

Trigger. three roots given, area between two of them; integrals given, areas asked (lobe bookkeeping). **Reflex.** name the lobes, sign them from the factors, solve the linear system. Never integrate a lobe you can name. *Seen:* 2018 P1 Q12

five points



5 points, unit 1

FORMULA

Left crossing, max, inflection, min, right crossing are equally spaced. A tangent at an extremum spans 3 units to its second crossing, cut $2:1$ by the other extremum. The inflection line meets the curve at $\pm\sqrt{3}$ units. Point symmetry about the inflection point: $\int$ between the turning points $=$ width $\times$ height of the inflection point.

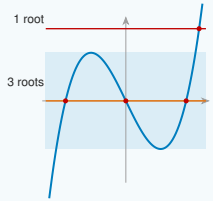
HOLDS WHEN

Any cubic with two turning points; unit $=$ inflection-to-extremum distance.

SUMMON WHEN

Trigger. cubic with a horizontal tangent; “turning points at α, β”; $\int_{\beta}^{\alpha} f$. **Reflex.** locate the five points before touching any formula. *Seen:* 2021 P1 Q2; 2018 P1 Q9

root band



FORMULA

Three distinct real roots $\iff$ $f(\text{max}) > 0 > f(\text{min})$: the axis passes strictly between the turning values. Exactly one real root $\iff$ the axis misses the band.

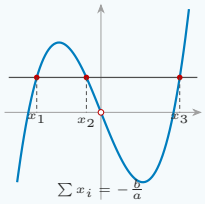
HOLDS WHEN

Cubic with two turning points: solve $f' = 0$, evaluate f at both.

SUMMON WHEN

Trigger. “three distinct real roots”; “exactly one real solution”; range of the constant. *Seen:* 2018 P1 Q9; 2017 P2 Q19; 2023 P1 Q14

root sum



FORMULA

Every line meeting the cubic keeps $\sum x_{\text{intersections}} = -b/a$. Only the x^3 and x^2 coefficients matter; c, d and the line's m, n do not.

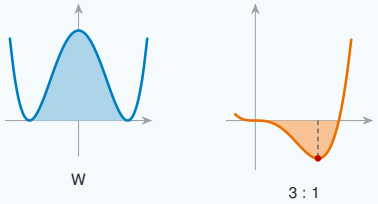
HOLDS WHEN

Count a tangency twice; it need not be at an extremum. Line and cubic must meet at least twice (else two roots are complex). Through the inflection point: sum $= 3\times$ inflection x.

SUMMON WHEN

Trigger. a line meets a cubic and one intersection is missing; two factors given, coefficients asked. *Seen:* 2020 P1 Q2

quartic lobes (F8 to F11)



FORMULA

W $a(x-\alpha)^2(x-\beta)^2$: middle hump $\frac{|a|}{30}\star^5$. $a(x-\alpha)^3(x-\beta)$: $\frac{|a|}{20}\star^5$. The extremum between α, β divides $[\alpha, \beta]$ $2:1$ for $a(x-\alpha)^2(x-\beta)$ (at $\alpha + \frac{2}{3}(\beta-\alpha)$) and $3:1$ for $a(x-\alpha)^3(x-\beta)$.

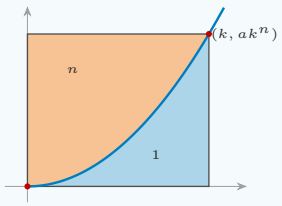
HOLDS WHEN

$\star = |\beta - \alpha|$. Optional: the general quartic-vs-line rule does not appear on TMUA/ESAT.

SUMMON WHEN

Trigger. repeated roots drawn; position of the extremum between two roots asked. *Seen:* NOIR questions

$n:1$ box



FORMULA

Rectangle from a stationary vertex (or inflection corner) to (k, ak^n) on $y = ax^n$: under the curve $= \dfrac{1}{n+1} \times$ rectangle, so above : below $= n : 1$. Parabola $2:1$, cubic $3:1$, quartic $4:1$. Nested chords on a parabola: doubling the width multiplies every area by 8 $(S, 8S, 64S)$.

HOLDS WHEN

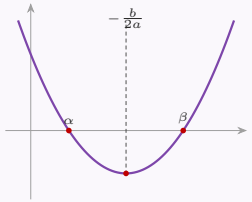
Independent of a and k; same for odd and even n. The box corner must be the stationary (or inflection) point.

SUMMON WHEN

Trigger. rectangle with a stationary point at a corner; shaded corner; $y = k\sqrt{x}$ (sideways parabola, vertex at the corner). *Seen:* 2019 P1 Q9

II. POLYNOMIALS (PURE ALGEBRA) 5 circuits

Vieta



FORMULA

$\sum$(products of k roots) $= (-1)^k a_{n-k}/a_n$. Quadratic: $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$. Cubic: $\sum r = -\frac{b}{a}$, $\sum r_i r_j = \frac{c}{a}$, $r_1 r_2 r_3 = -\frac{d}{a}$.
$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$; $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$;
$(\sqrt{\alpha} + \sqrt{\beta})^2 = \alpha + \beta + 2\sqrt{\alpha\beta}$.

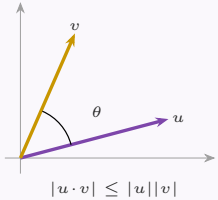
HOLDS WHEN

Roots counted with multiplicity, complex ones included; $a_n \neq 0$.

SUMMON WHEN

Trigger. "sum of the values of x"; "roots differ by"; symmetric function of the roots; $(x - \sqrt{\alpha})(x - \sqrt{\beta})$. **Reflex.** never solve for the roots. *Seen:* Spec P1 Q1, Q9; 2020 P1 Q9

Cauchy-Schwarz



FORMULA

$\left(\sum a_k^2\right)\left(\sum x_k^2\right) \geq \left(\sum a_k x_k\right)^2$; $(a^2 + b^2)(x^2 + y^2) \geq (ax + by)^2$.
Equality iff the two lists are proportional.

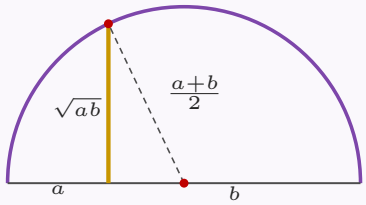
HOLDS WHEN

All reals. Proof: dump to one side, complete squares, $(ay - bx)^2 \geq 0$.

SUMMON WHEN

Trigger. $x^2 + y^2 =$ const, max or min of $px + qy$; "least value of ab given a product identity". Backup circuit: not yet a named TMUA engine, do not lead with it. *Seen:* 2022 P1 Q13 flavour

AM-GM



FORMULA

$\frac{1}{n}(a_1 + \cdots + a_n) \geq \sqrt[n]{a_1 \cdots a_n}$, equality iff all equal. $x + \frac{k}{x} \geq 2\sqrt{k}$ for $x > 0$.

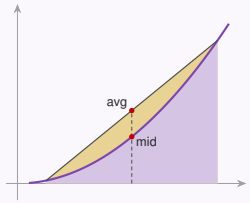
HOLDS WHEN

Non-negative terms only. Visual: on a semicircle, radius $\frac{a+b}{2} \geq$ altitude $\sqrt{ab}$.

SUMMON WHEN

Trigger. minimise a sum containing a term and its reciprocal on a positive domain. **Reflex.** split into AM-GM-ready pieces; the minimum is the equality case.

midpoint chord



FORMULA

Concave up ($f'' \geq 0$): $\frac{1}{2}\big(f(a)+f(b)\big) \geq f\big(\frac{a+b}{2}\big)$, chord above curve, trapezium rule overestimates. Concave down: reversed, underestimates.

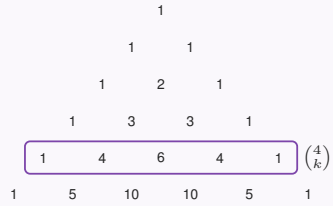
HOLDS WHEN

Interval of fixed concavity. Adding a constant, positive scaling, sliding, reflecting in the y-axis keep concavity; $1-f$ (or $-f$) flips it.

SUMMON WHEN

Trigger. "necessary condition for $\frac{f(a)+f(b)}{2} \geq f(\frac{a+b}{2})$"; "trapezium rule over/underestimates"; which transformed integral flips the estimate. Never draw the trapezia.
Seen: 2018 P2 Q18; 2019 P1 Q8; 2020 P2 Q12

multinomial



FORMULA

$(a+bx)^n = \sum_k \binom{n}{k} a^{n-k}(bx)^k$, $\binom{n}{r} = \binom{n}{n-r}$. Coefficient of $x^p y^q$ in $(Ax+By+C)^n$: $\frac{n!}{p!\,q!\,(n-p-q)!}A^pB^qC^{n-p-q}$. In $\big(\sum a_k x^k\big)^n$ with $a_k x^k$ chosen n_k times: $\sum n_k = n$, power of $x = \sum k\,n_k$.

HOLDS WHEN

Exponent budget conserved. Constant term: total power $= 0$, solve for k. Sum of all coefficients $= f(1)$.

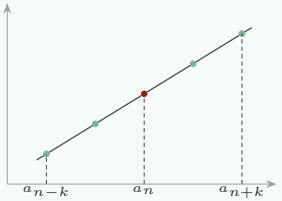
SUMMON WHEN

Trigger. "coefficient of"; "constant term"; $\sum_k (1+x)^k$ $(\sum \binom{k}{1} = \frac{m(m+1)}{2})$. **Reflex.** isolate the one term that matters; never expand. *Seen:* 2020 P1 Q13; Spec P1 Q20; 2018 P2 Q2; 2019 P1 Q3

III. SEQUENCES / SERIES

4 circuits

double the middle



FORMULA

AP: $2a_{n+1} = a_n + a_{n+2}$; $2a_n = a_{n-k} + a_{n+k}$; $a_n - a_{n-k} = kd$; $a_n = a_1 + (n-1)d$. A run of terms symmetric about a centre sums to (count) $\times$ (centre).

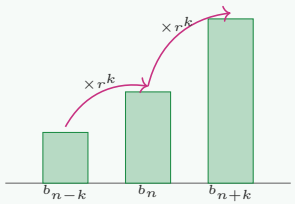
HOLDS WHEN

Terms equally spaced, not necessarily consecutive; a_0, a_{-1} allowed. AP $\subset$ linear functions, gradient d.

SUMMON WHEN

Trigger. three AP terms given as expressions; mixed AP/GP conditions; "$S_5 = S_8$" $(a_6 + a_7 + a_8 = 0 \Rightarrow a_7 = 0)$. *Seen:* 2017 P1 Q7; 2018 P1 Q2; Spec P1 Q19

square the middle



FORMULA

GP: $b_{n+1}^2 = b_n b_{n+2}$; $b_n^2 = b_{n-k}b_{n+k}$; $b_{n+k}/b_n = r^k$. A block of terms k places later $=$ the earlier block $\times r^k$. Squares of a GP form a GP with ratio r^2, cubes r^3.

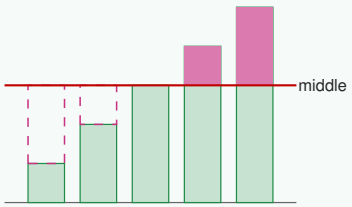
HOLDS WHEN

Terms equally spaced; $r \neq 0$. GP $\subset$ exponential functions, base r. Squaring, not doubling: the distractor swaps the two.

SUMMON WHEN

Trigger. "$S_6 = 9S_3$" $(1+r^3 = 9)$; "$S_{30} - S_{20} = kS_{10}$" $(k = r^{20})$; sum to infinity of squares or cubes given.
Seen: 2019 P1 Q5; 2022 P1 Q8; 2018 P1 Q8

middle term



FORMULA

$S_n = n \times$ middle term ($a_{(n+1)/2}$; for even n, the mean of the two central terms). $S_n = \frac{n}{2}\big(2a + (n-1)d\big)$. GP: $S_n = \frac{a(r^n-1)}{r-1}$, $S_\infty = \frac{a}{1-r}$ $(|r| < 1)$.

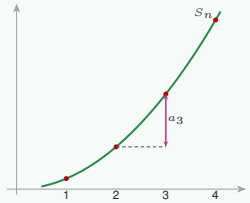
HOLDS WHEN

Middle-term rule is for APs.

SUMMON WHEN

Trigger. "sum of the first m terms equals the sum of the first $2m$": terms $m+1$ to $2m$ sum to 0, so their middle is 0. *Seen:* 2020 P2 Q14

truncate and integrate



FORMULA

$a_n = pn + q \Rightarrow S_n$: keep pn, integrate to $\frac{p}{2}n^2$, fix the n-coefficient with $S_1 = a_1$. $a_n = S_n - S_{n-1}$ $(n \geq 2)$, $a_1 = S_1$. Sum of coefficients of $a_n =$ sum of coefficients of S_n.

HOLDS WHEN

AP from the first term $\iff S_n$ is a quadratic in n with no constant term; a constant term means arithmetic only from $n = 2$.

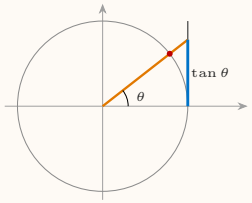
SUMMON WHEN

Trigger. S_n given as a polynomial; "is the sequence arithmetic"; sequences hiding inside integrals ($\int_n^{n+1} f = n + 1$; $\int_0^{99} 2^{\lceil x \rceil}$). *Seen:* 2021 P1 Q13; 2023 P2 Q17

IV. TRIGONOMETRY

10 circuits

$\tan\theta$



FORMULA

Gradient of a line $= \tan\theta$, θ its angle with the positive x-axis. On the unit circle $P(\cos\theta, \sin\theta)$: coordinates with signs, not just lengths.

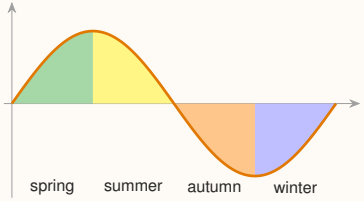
HOLDS WHEN

Any non-vertical line; the angle is the same against every horizontal line.

SUMMON WHEN

Trigger. "acute angle between the line and the x-axis"; tangent to two equal circles (internal tangents pass through the midpoint of the centres). *Seen:* 2020 P1 Q16

seasons



FORMULA

Think in quarters. Steps of $90°$ are steps of one season; four consecutive terms cancel: $\sin\theta + \sin(\theta + 90°) + \sin(\theta + 180°) + \sin(\theta + 270°) = 0$. $\sin^2 k + \sin^2(90° - k) = 1$.

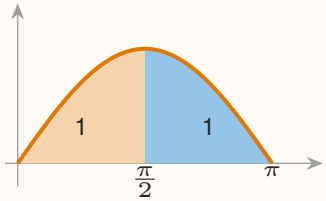
HOLDS WHEN

Sine starts in spring (rises from 0), cosine in summer (falls from 1); one hump in four quarters.

SUMMON WHEN

Trigger. $\sum \sin(10° + 90°k)$; $\sum_0^{90} \sin^2 k°$. **Reflex.** count full cycles, keep the remainder; pair k with $90 - k$, the middle term is unpaired. *Seen:* 2019 P1 Q19; 2018 P1 Q20

season area



FORMULA

One quarter of $y = a\sin bx$ has area $|a|/|b|$; every block of $\sin x$ or $\cos x$ has area 1. Period of $a\sin(bx+c)+d$ is $\frac{2\pi}{|b|}$; of $a\tan(bx+c)+d$ is $\frac{\pi}{|b|}$.

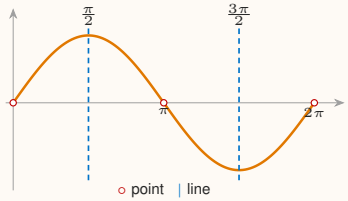
HOLDS WHEN

Stretch in x by k: area $\times k$; stretch in y by k: area $\times k$. Any periodic zigzag too: squeezing by 2^{r-1} leaves $\int_0^1$ unchanged.

SUMMON WHEN

Trigger. area under a stretched sine over a quarter or half cycle; $\int_0^1 \sum_r f(2^{r-1}x)$. *Seen:* 2021 P1 Q15

trig symmetry



FORMULA

sin: point-symmetric about $(n\pi, 0)$, line-symmetric about $x = \frac{2k+1}{2}\pi$. cos: line-symmetric about $x = n\pi$, point-symmetric about $(\frac{2k+1}{2}\pi, 0)$. Line of symmetry of $a\sin(bx+c)$: argument $= \frac{\pi}{2} + n\pi$ (through the peaks). Translate the axes, not the function: shift f by $(+2, -3)$ = push the y-axis left 2, the x-axis up 3.

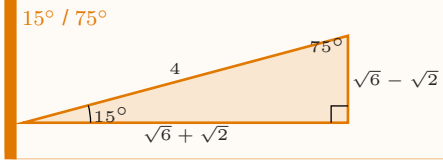
HOLDS WHEN

Every $n, k \in \mathbb{Z}$. Sine and cosine are one curve slid by $\frac{\pi}{2}$.

SUMMON WHEN

Trigger. "smallest positive a for which $x = a$ is a line of symmetry of $\sin(2x - \frac{4\pi}{3})$". *Seen:* 2018 P1 Q18

$15° / 75°$



FORMULA

$\sin 15° = \cos 75° = \frac{\sqrt{6}-\sqrt{2}}{4}$, $\cos 15° = \sin 75° = \frac{\sqrt{6}+\sqrt{2}}{4}$, $\tan 15° = 2-\sqrt{3}$, $\tan 75° = 2+\sqrt{3}$. Right triangle: hypotenuse 4, legs $\sqrt{6} \mp \sqrt{2}$.

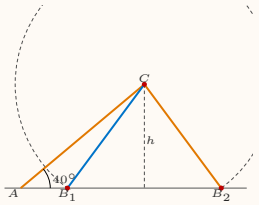
HOLDS WHEN

From $15° = 45° - 30°$ or by splitting the equilateral triangle.

SUMMON WHEN

Trigger. $\sqrt{6} \pm \sqrt{2}$, $2 \pm \sqrt{3}$ in the options; $15°, 75°, 105°$ inside squares and quarter circles ($\sqrt{2}-1$, $\sqrt{4-2\sqrt{2}}$, $2\sqrt{2}-2$). *Seen:* visual plates

ambiguous case



FORMULA

SSA: drop the perpendicular first. $h = (\text{given side})\sin(\text{given angle})$; the other side p closes on either side of the foot at $\pm\sqrt{p^2-h^2}$. $p = h$: one triangle; $h < p <$ given side: two; $p \geq$ given side: one. Same height, so areas are in the ratio of the bases.

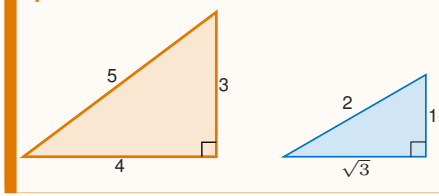
HOLDS WHEN

Two sides and a non-included angle.

SUMMON WHEN

Trigger. "two distinct triangles"; "for which p is PQ uniquely determined"; "the larger has three times the area of the smaller". *Seen:* 2023 P1 Q8; 2018 P2 Q14; 2018 P1 Q19

special ratios



FORMULA

$3:4:5$, $5:12:13$, $8:15:17$, $7:24:25$ and multiples; $3:5:7$ with $120°$ between 3 and 5; $3:7:8$ with $60°$. Altitude 12 splits $15:20:25$ into $9:12:15$ and $12:16:20$. Equilateral side x: height $\frac{\sqrt{3}}{2}x$, area $\frac{\sqrt{3}}{4}x^2$, centroid cuts medians $2:1$; 30-60-90 is $1:\sqrt{3}:2$.

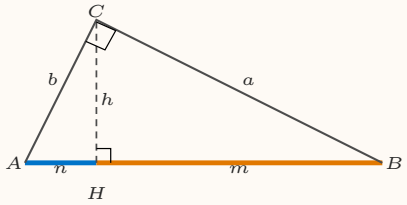
HOLDS WHEN

Read lengths, never Pythagoras. Everything is $\sqrt{2}$ when everything is $45°$.

SUMMON WHEN

Trigger. tangent from $(20, 0)$ to radius 12 (tangent 16, $3:4:5$ scaled); centres 9 across and 9 up $(9\sqrt{2})$; prism with equilateral cross-section. *Seen:* 2017 P1 Q6; 2018 P1 Q3; Spec P1 Q12

$h^2 = mn$



FORMULA

Right angle at C, altitude $CH = h$ to the hypotenuse, $BH = m$, $HA = n$: $ab = ch$, $a^2 = mc$, $b^2 = nc$, $h^2 = mn$. Triangles HBC, HCA, CBA are similar.

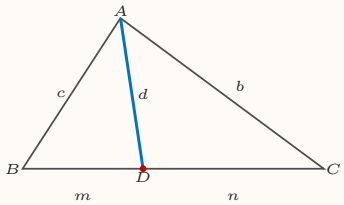
HOLDS WHEN

Right triangle only; $h^2 = mn$ is iff for the right angle. Each relation sprouts from one vertex (C, B, A, H).

SUMMON WHEN

Trigger. kite or rhombus (perpendicular diagonals); "necessary and sufficient for $\angle SPQ = 90°$" gives $x^2 = yz$. *Seen:* 2022 P2 Q11

Apollonius, Stewart, Ceva, Menelaus



FORMULA

Median $AM = d$, $BM = MC = m$: $b^2 + c^2 = 2(m^2 + d^2)$. Cevian $AD = d$, $BD = m$, $DC = n$: $mb^2 + nc^2 = (m + n)(d^2 + mn)$. Ceva (concurrent cevians): $\frac{b}{a} \cdot \frac{d}{c} \cdot \frac{f}{e} = 1$ around the triangle. Menelaus (one transversal): the same product $= 1$.

HOLDS WHEN

Any triangle, no angle conditions. Ceva: a point inside; Menelaus: two triangles sharing an angle.

SUMMON WHEN

Trigger. median or cevian length; ratio chase along a transversal. Optional for TMUA.

geometric shading



FORMULA

Nested similar shapes: first term a, side ratio r, total shaded area $= a/(1 - r^2)$. Regular polygon: n equidistant points on a circle, central angles in an AP summing to 2π; skipping k points gives equal chords.

HOLDS WHEN

Every stage similar to the last. Find the side ratio, never the area ratio, then square it.

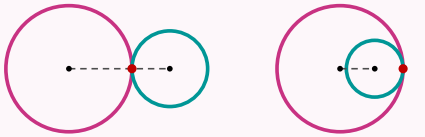
SUMMON WHEN

Trigger. recursive figure; "shaded area after infinitely many steps"; circles C_n with radii in a sequence (pair them: $2\pi\big((2k)^2 - (2k-1)^2\big)$). *Seen:* 2023 P1 Q17

V. GEOMETRY (CIRCLES)

5 circuits

tangency



$d = R + r$ $d = R - r$

FORMULA

Centres d apart, radii $R > r$: externally tangent $d = R + r$, internally $d = R - r$. Common tangent length: external $\sqrt{d^2 - (R - r)^2}$, internal $\sqrt{d^2 - (R + r)^2}$.

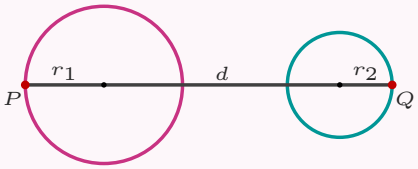
HOLDS WHEN

"Exactly one point in common" means tangency, two cases, two answers.

SUMMON WHEN

Trigger. two circle equations with one common point, both r asked; length of a common tangent. *Seen:* 2019 P1 Q6

line of centres



$\max PQ = d + r_1 + r_2$

FORMULA

Farthest pair of points on two circles: on the line of centres, $\max PQ = d + r_1 + r_2$. Farthest point on a circle from a chord PQ: on the far side, height $R + R\cos\frac{1}{2}\angle POQ$; chord $= 2R\sin\frac{1}{2}\angle POQ$. A line bisects a circle (or square) only through its centre. Tangent to the y-axis $\iff$ $|x_{\text{centre}}| =$ radius.

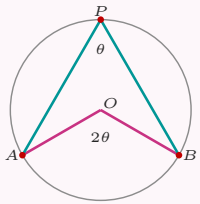
HOLDS WHEN

Complete the square first. For $x^2 + ax + y^2 + by + c = 0$, $a, b, c \neq 0$: tangent to the y-axis $\iff$ $b^2 = 4c$.

SUMMON WHEN

Trigger. "maximum possible length of PQ"; "greatest possible area of triangle PRQ"; "bisects both the circle and the square"; "tangent to the y-axis". *Seen:* 2023 P1 Q13; 2022 P1 Q14; 2021 P2 Q7; 2021 P2 Q15

inscribed angle



FORMULA

$\angle AOB = 2\angle APB$ for any P on the remaining arc; holds for major arcs; angles on the same arc are equal; angle in a semicircle $= 90°$. Minor arc: central $\leq 180°$, inscribed $\leq 90°$; major arc the reverse.

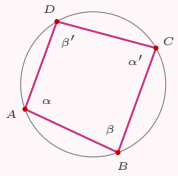
HOLDS WHEN

AB need not be a diameter; OP need not be perpendicular to anything. Vocabulary: arc, chord, secant (extended chord), tangent, normal, sector, segment.

SUMMON WHEN

Trigger. any circle angle chase, often stacked two or three deep; $\angle OAB$ given (isosceles OAB first, then halve). *Seen:* 2022 P1 Q14; 2023 P1 Q13

cyclic quad



$\alpha + \alpha' = \beta + \beta' = 180°$

FORMULA

$\angle A + \angle C = 180°$, $\angle B + \angle D = 180°$. Exterior angle at a vertex $=$ interior angle at the opposite vertex.

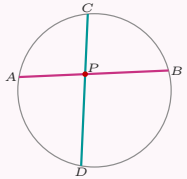
HOLDS WHEN

Four vertices on one circle; nothing else.

SUMMON WHEN

Trigger. a side extended with the exterior angle marked; "not enough information" when the quadrilateral is not known to be cyclic (a trapezium changes the answer). *Seen:* NOIR configurations

power of a point



$PA \cdot PB = PC \cdot PD$

FORMULA

Chords through interior P: $PA \cdot PB = PC \cdot PD$. Secants from exterior P: $PA \cdot PC = PB \cdot PD$ (near $\times$ far on each line). Tangent PQ: $PQ^2 = PA \cdot PB$ for every line from P.

HOLDS WHEN

P fixed, any two lines through it. Proved by similar triangles; never re-derive it live.

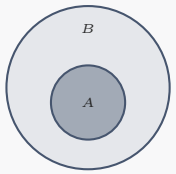
SUMMON WHEN

Trigger. two chords crossing; tangent and secant from one point; "find the missing length". The skill tested is choosing chord-chord vs tangent-secant. *Seen:* NOIR questions

VI. LOGIC

3 circuits

larger set



$A \Rightarrow B$: $A \subset B$

FORMULA

$A \Rightarrow B \equiv B$ if $A \equiv A$ sufficient for $B \equiv B$ necessary for $A \equiv A \subset B$. "X only if Y" is $X \Rightarrow Y$. Converse ($B \Rightarrow A$) and inverse ($\neg A \Rightarrow \neg B$) are not equivalent to it; the contrapositive ($\neg B \Rightarrow \neg A$) is.

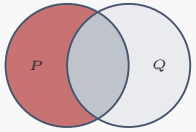
HOLDS WHEN

Mission statement: "what is the larger set?" Sufficient $=$ smaller set; necessary $=$ larger set. Never reason from the English; plug in values.

SUMMON WHEN

Trigger. "necessary but not sufficient"; "which are equivalent to the theorem"; yes/no tables. **Reflex.** translate every clause to an arrow, test with one number. *Seen:* 2019 P2 Q5; 2023 P2 Q11

counterexample



$P \setminus Q \neq \emptyset$

FORMULA

A counterexample to $p \Rightarrow q$ has p true AND q false, nothing else: $P \setminus Q \neq \emptyset$.

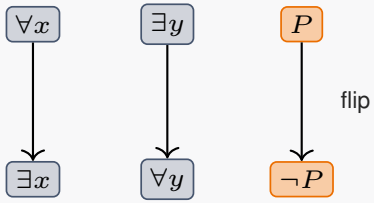
HOLDS WHEN

Cases with p false prove nothing either way.

SUMMON WHEN

Trigger. "which of the following are counterexamples". *Seen:* 2018 P2 Q3

compart-mentalise and negate



FORMULA

Chunk the statement, negate each chunk: $\forall \leftrightarrow \exists$ in order, then flip the final claim. $\neg(A \wedge B) = \neg A \vee \neg B$; $\neg(A \Rightarrow B) = A \wedge \neg B$.

HOLDS WHEN

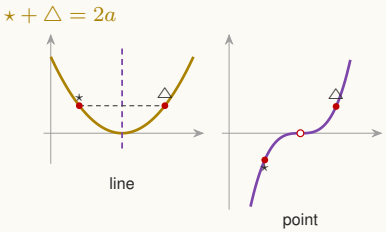
Every quantifier flips; the final claim flips once. Mechanically, in full.

SUMMON WHEN

Trigger. "the negation of"; "if (*) is not true, which is certainly true". Failure mode: negating one clause, leaving a quantifier untouched. *Seen:* 2018 P2 Q12; Spec P2 Q9

VII. FUNCTION TRANSFORMATION

3 circuits



FORMULA

Line symmetry about $x = a$: $f(\star) = f(\triangle)$ whenever $\star + \triangle = 2a$ [$f(a + x) = f(a - x)$; $f(x) = f(2a - x)$]. Point symmetry about (a, b): $f(\star) + f(\triangle) = 2b$ whenever $\star + \triangle = 2a$. Even $=$ line symmetry about $x = 0$; odd $=$ point symmetry about $(0, 0)$.

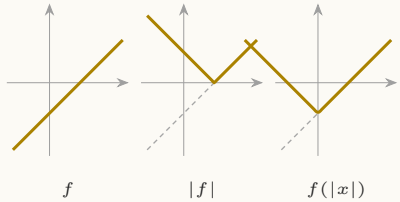
HOLDS WHEN

Add the arguments: constant sum $2a$ gives the axis or centre. Outputs equal: a line. Outputs summing to a constant: a point. Arguments differing by a constant ($f(x) = f(x + 4)$): a period, not a symmetry.

SUMMON WHEN

Trigger. "$f(x) + f(6 - x) = 10$"; "sufficient condition for $\int_1^3 f = 0$" (point symmetry about $(2, 0)$ kills $\int_{2-t}^{2+t}$); "reflect then translate equals translate then reflect" (period). *Seen:* 2018 P2 Q10; 2017 P2 Q10; 2019 P2 Q20

modulus family



FORMULA

$f(-x)$: reflect in the y-axis. $-f(x)$: flip in the x-axis. $f(|x|)$: right half mirrored to the left. $|f(x)|$: negative parts reflected up. $|y| = f(x)$: discard $y < 0$, mirror the rest down. $|y| = |f(x)|$: $y = \pm f(x)$. $|y| = f(|x|)$: first quadrant copied into all four.

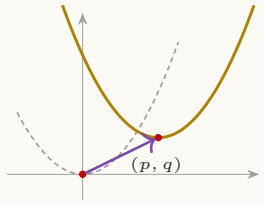
HOLDS WHEN

Read nested moduli inside out; order matters.

SUMMON WHEN

Trigger. sketch matching with moduli; counting solutions of $|f(x)| = k$.

vertex tracking



FORMULA

Move only the vertex and the leading coefficient. Translation (p, q): vertex $+(p, q)$. Reflection in the x-axis: $a \to -a$. Reflection in $y = 1$: $y \to 2 - y$. Stretch parallel to x by k: vertex $x \times k$, quadratic coefficient $\div k^2$. $f(kx)$ squeezes the vertex to x/k; $f(x - c)$ slides it to $x + c$; both keep the extreme value.

HOLDS WHEN

Any single-vertex curve; rebuild $y = a(x - h)^2 + k$ last. Order matters.

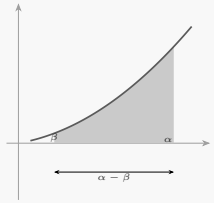
SUMMON WHEN

Trigger. "apply (1), (2), (3) to $y = 4x^2$"; "$f(kx)$ and $f(x - c)$ have the same minimum point"; $\cos x$ reflected in $y = 1$ then translated. *Seen:* 2020 P1 Q10; Spec P1 Q10; 2023 P1 Q11

+ AFTER "MATH"

1 circuits

group then subtract



FORMULA

$\left[ax^2 + bx\right]_\beta^\alpha = (\alpha - \beta)\big(a(\alpha + \beta) + b\big)$. In general

$\left[\sum a_k x^k\right]_\alpha^\beta = \sum a_k\big(\beta^k - \alpha^k\big)$: factorise each difference, then multiply by its coefficient.

HOLDS WHEN

Every definite integral of a polynomial or power. Never evaluate the two brackets separately.

SUMMON WHEN

Trigger. any $\int_\alpha^\beta$ of a polynomial; $\int_1^4 \big(3\sqrt{x} + 4x^{-2}\big)dx$.
Reflex. group same powers, subtract inside, take out the coefficient. *Seen:* 2021 P2 Q1; 2019 P1 Q16