# DAWN PUTNAM SESSION

Putnam 1993 — Papers A and B

Putnam 120 Archive

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# Putnam 1993 — Paper A

## 1993 A1

**1993 A1.** The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find $c$ so that the areas of the two shaded regions are equal. [Figure not included. The first region is bounded by the $y$-axis, the line $y = c$ and the curve; the other lies under the curve and above the line $y = c$ between their two points of intersection.]

**Fields:** calculus (definite integrals as signed area), polynomial algebra **Position:** A1 – opening problem of the first session **Answer:** $c = \frac{4}{9}$.

**Layer 1 THE WHAT — surface analysis**

Throughout, write $f(x) = 2x - 3x^3$. On $x \geq 0$ the curve starts at the origin, rises to its maximum at $f'(x) = 2 - 9x^2 = 0$, i.e. $x = \frac{\sqrt{2}}{3}$, where $f = \frac{4\sqrt{2}}{9} \approx 0.629$, and falls back to 0 at $x = \sqrt{2/3}$. For $0 < c < \frac{4\sqrt{2}}{9}$ the line $y = c$ meets this arch at two points $0 < a < b$. The first region is $\{0 \leq x \leq a,\ f(x) \leq y \leq c\}$; the second is $\{a \leq x \leq b,\ c \leq y \leq f(x)\}$.

*The natural attempt.* You translate the picture straight into two integrals and set them equal:

$$\int_0^a \big(c - f(x)\big)\,dx \;=\; \int_a^b \big(f(x) - c\big)\,dx.$$

To evaluate either side you need $a$ and $b$, and they are the positive roots of $3x^3 - 2x + c = 0$. So you try to solve that cubic for $x$ in terms of $c$. It has no rational root for general $c$; the discriminant route leads to Cardano's formula, or to the trigonometric form $x = \frac{2\sqrt{2}}{3}\cos\theta$ with $\cos 3\theta$ expressed in $c$. You now have $a$ and $b$ as $\frac{2\sqrt{2}}{3}\cos\big(\frac{1}{3}\arccos(\cdot) \pm \cdots\big)$, and the equation you must solve for $c$ is a polynomial in two nested trigonometric expressions. This is the wall. Every line of it is correct, and none of it is going to finish inside the time a first problem deserves. (A student who persists sometimes tries to guess $c$ from a sketch, which gives a number, not a proof.)

*The model chain.* Subtract the left side from the right. Equality of the two areas is equivalent to

$$\int_a^b \big(f(x) - c\big)\,dx - \int_0^a \big(c - f(x)\big)\,dx = 0 \iff \int_0^b \big(f(x) - c\big)\,dx = 0. \qquad (*)$$

The inner intersection point $a$ has disappeared. Now use $b$, not $c$, as the unknown: $b$ lies on the curve, so $c = f(b) = 2b - 3b^3$. Evaluate $(*)$:

$$\int_0^b (2x - 3x^3)\,dx - cb = b^2 - \tfrac{3}{4}b^4 - b(2b - 3b^3) = \tfrac{9}{4}b^4 - b^2 = b^2\big(\tfrac{9}{4}b^2 - 1\big).$$

With $b > 0$ this vanishes only at $b = \frac{2}{3}$, giving $c = f(\frac{2}{3}) = \frac{4}{3} - \frac{8}{9} = \frac{4}{9}$.

*Validation.* The reduction used that $b$ is the *right-hand* intersection, so check the configuration. Factor

$$2x - 3x^3 - \tfrac{4}{9} = \big(\tfrac{2}{3} - x\big)\big(3x^2 + 2x - \tfrac{2}{3}\big),$$

whose other roots are $x = \frac{-1\pm\sqrt{3}}{3}$. One is negative; the other is $a = \frac{\sqrt{3}-1}{3} \approx 0.244$, which lies in $(0, \frac{2}{3})$. So on $x \geq 0$ the line $y = \frac{4}{9}$ meets the curve exactly at $a$ and $b = \frac{2}{3}$, with $f < \frac{4}{9}$ on $[0, a)$ and $f > \frac{4}{9}$ on $(a, b)$ – the figure's configuration, with $0 < \frac{4}{9} < \frac{4\sqrt{2}}{9}$. (Numerically, each region has area $\approx 0.05157$.) Hence $c = \frac{4}{9}$.

**Layer 2 THE HOW — the bifurcation**

**Bifurcating point.** Both chains agree on the equation of areas with limits $0$, $a$, $b$. They split at the very next line. The natural attempt says "I need $a$ and $b$ in terms of $c$" and goes to the cubic. The model says "one area is the other with a sign flip, so move it across" and obtains $(*)$, in which $a$ has vanished; then it takes $b$ as the unknown and reads $c$ off as $f(b)$.

> *Equal areas, one above the line and one below it. If I integrate $f - c$ all the way from $0$ to $b$, the region below counts negative and the region above counts positive, so "equal areas" is "that integral is zero". The awkward inner root drops out.*

> *I cannot solve the cubic for $x$ in terms of $c$, but I do not have to: $c$ is a polynomial in $b$, because $b$ is on the curve. Make $b$ the unknown and substitute $c = 2b - 3b^3$.*

**Layer 3 THE WHY**

Kedlaya's solution opens with "we wish to find $c$ such that $\int_0^b \big((2x - 3x^3) - c\big)dx = 0$" and gives no reason for that integral. Here is where it comes from.

The definite integral is, by definition, *signed* area: $\int_p^q g$ counts the region where $g > 0$ positively and where $g < 0$ negatively. The two shaded regions are exactly the regions between $y = f(x)$ and $y = c$ on $[0, b]$, one with $f - c < 0$ and one with $f - c > 0$. So the phrase "the two areas are equal" is, word for word, the statement that the positive and negative contributions of $f - c$ on $[0, b]$ cancel. The problem is not asking you to compute two areas; it is asking whether one signed integral is zero. Once you see equal-and-opposite pieces of area, gluing them into one integral is not a trick – it is reading the definition backwards. And the point $a$ where the sign changes is precisely the thing a signed integral never needs to know: that is what the signed integral is for.

The second move – parametrise by $b$ – is the same principle in algebra. The relation between $b$ and $c$ is $c = f(b)$. In one direction it is an explicit polynomial; in the other it is the inverse of a cubic, which has no usable closed form. Whenever two unknowns are tied by an equation that is explicit one way and implicit the other, choose as your variable the one that makes the other explicit. You never invert what you can instead evaluate. The problem's phrasing even points at it: the second region ends "between their two points of intersection", so the right-hand intersection is a named object on the figure, a legitimate thing to call $b$.

There is also a way to see the answer's shape before computing. Condition $(*)$ says $\int_0^b f = b\,f(b)$: the average value of $f$ over $[0, b]$ equals its value at the endpoint $b$. For $f(x) = 2x - 3x^3$ the average over $[0, b]$ is $b - \frac{3}{4}b^3$, and setting this equal to $2b - 3b^3$ is a one-line equation. The average-value reading makes clear why a unique $b$ exists: the average starts below the endpoint value (both are near $0$ but the endpoint value grows like $2b$ against $b$) and the curve eventually falls below its own running average.

Finally, the validation is not decoration. $(*)$ is necessary and sufficient only if $b$ really is the rightmost intersection and $a$ really lies between $0$ and $b$. Solving $(*)$ assumes the picture; the factorisation confirms the picture. Omitting that check is where a correct answer becomes an incomplete solution.

**Layer $+\alpha$ BACKTESTING**

**1.** The first useful cue is "the areas of the two shaded regions are equal", with one region under

the line and one over it. On seeing it, write $\int_0^b (f(x) - c)\,dx = 0$, with $b$ the right-hand intersection.

**2.** It suggests signed area: merge the two area integrals into a single definite integral whose value is zero, which eliminates the inner intersection point.

**3.** Hesitation occurs when you realise $a$ and $b$ are roots of a cubic in $c$ and reach for Cardano. What reinforces it away is the question "which unknown makes the others explicit?" – here $b$, since $c = f(b)$.

**4.** The trap is the literal translation: two separate integrals whose limits are roots of a cubic. It is correct and unfinishable. A secondary trap is forgetting to check that $b = \frac{2}{3}$ is the rightmost intersection and that the configuration matches the figure.

**5.** Recognise faster by treating "equal areas on opposite sides of a line" as the phrase "net signed area zero", and by noticing whenever a parameter is determined by a point on a curve – that point is the better variable.

**6.** The reusable rule: when a condition compares quantities of opposite sign, combine them into one signed quantity and set it to zero; when two unknowns are linked by a relation explicit in one direction only, choose the variable that makes the relation explicit.

▷ **Cue:** two regions between a curve and a line, one above and one below, are required to have equal area $\longrightarrow$ **Move:** set the single signed integral of (curve minus line) over the combined interval equal to zero; the inner crossing point cancels

▷ **Cue:** integration limits are roots of a polynomial in the parameter you are asked to find $\longrightarrow$ **Move:** do not solve for the roots; make a limit the unknown and express the parameter as a function of it

▷ **Cue:** a condition of the form $\int_0^b g = b\,g(b)$ appears $\longrightarrow$ **Move:** read it as "average value on $[0, b]$ equals endpoint value"

▷ **Cue:** an answer was found by assuming the configuration in a figure $\longrightarrow$ **Move:** factor out the known root and check the remaining roots sit where the figure says

## 1993 A2

**1993 A2.** Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove there exists a real number $a$ such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

**Fields:** algebra (recurrences, invariants) **Position:** A2 – second problem of the morning session **Answer:** Take $a = (x_0 + x_2)/x_1$; subtracting consecutive instances of the hypothesis shows $(x_{n+1} + x_{n-1})/x_n$ does not depend on $n$, so it always equals $a$.

### Layer 1 THE WHAT — surface analysis

*The natural attempt.* You read "prove there exists $a$" and decide to find $a$ first. The target $x_{n+1} = ax_n - x_{n-1}$ at $n = 1$ reads $x_2 = ax_1 - x_0$, so $a$ has to be $(x_0 + x_2)/x_1$ – there is no choice. Now you try to prove the recurrence for $n = 2$ by brute force. From the hypothesis at $n = 2$, $x_3 = (x_2^2 - 1)/x_1$. You want this to equal $ax_2 - x_1 = \frac{(x_0+x_2)x_2}{x_1} - x_1$. Multiply by $x_1$: you need $x_2^2 - 1 = x_0x_2 + x_2^2 - x_1^2$, that is $x_1^2 - x_0x_2 = 1$, which is the hypothesis at $n = 1$. It works. Then you set up an induction: assume $x_{k+1} = ax_k - x_{k-1}$ for all $k \leq n$, substitute, expand $x_{n+1}^2 - 1$ in terms of $a$, $x_n$, $x_{n-1}$, and try to reach $x_n(ax_{n+1} - x_n)$. This does go through (it is Solution 2 in Kedlaya–Poonen–Vakil), but each step is a small expansion in three quantities, and it is easy to lose track of which instance of the hypothesis you are using where. You get there, having done more algebra than the problem needed.

*The model chain.* For $n \geq 1$ write $D_n = x_n^2 - x_{n-1}x_{n+1}$; the hypothesis is $D_n = 1$ for all $n \geq 1$.

*Compare two consecutive instances.* For $n \geq 1$, $D_n = D_{n+1}$, i.e.

$$x_n^2 - x_{n-1}x_{n+1} = x_{n+1}^2 - x_n x_{n+2}.$$

Move the terms so each side has a common factor:

$$x_n^2 + x_n x_{n+2} = x_{n+1}^2 + x_{n-1}x_{n+1}, \qquad \text{i.e.} \qquad x_n\,(x_n + x_{n+2}) = x_{n+1}\,(x_{n+1} + x_{n-1}).$$

*Divide.* Since $x_n \neq 0$ and $x_{n+1} \neq 0$, divide by $x_n x_{n+1}$:

$$\frac{x_{n+2} + x_n}{x_{n+1}} = \frac{x_{n+1} + x_{n-1}}{x_n}.$$

So with $a_n = (x_{n+1} + x_{n-1})/x_n$ for $n \geq 1$, we have $a_{n+1} = a_n$ for every $n \geq 1$.

*Conclude.* By induction $a_n = a_1$ for all $n \geq 1$. Put $a = a_1 = (x_0 + x_2)/x_1$, a real number. Then for every $n \geq 1$, $(x_{n+1} + x_{n-1})/x_n = a$, which rearranges to $x_{n+1} = ax_n - x_{n-1}$. □

**Layer 2 THE HOW — the bifurcation**

**Bifurcating point.** Both chains agree on the target: the only possible $a$ is $(x_{n+1} + x_{n-1})/x_n$ read off at $n = 1$. They split at the next line. The natural attempt fixes $a$ from the first three terms and pushes the recurrence forward term by term, substituting. The model rewrites the target as "the ratio $(x_{n+1} + x_{n-1})/x_n$ is the same for every $n$" and proves that by setting two consecutive instances of the hypothesis equal.

> *The problem wants a constant $a$ with $x_{n+1} + x_{n-1} = ax_n$. So what it is really asking is: is $(x_{n+1} + x_{n-1})/x_n$ independent of $n$? To show something doesn't depend on $n$, I show step $n$ and step $n+1$ give the same value.*

> *The hypothesis says one expression equals 1 for every $n$. The only way to use "for every $n$" is to compare two different $n$. I set the $n$ and $n+1$ instances equal and see what cancels.*

**Layer 3 THE WHY**

Why is "set consecutive instances equal" the natural move rather than an inspired one? Because of what the hypothesis actually says. The right-hand side is the constant 1; the number itself is irrelevant (replace 1 by any constant and the proof is unchanged). What carries information is that $x_n^2 - x_{n-1}x_{n+1}$ is *the same* for all $n$ – it is an invariant. An invariant used once is a single equation among three unknowns; an invariant used at two adjacent indices gives a relation that involves four consecutive terms, which is exactly the length of window the conclusion talks about ($x_{n-1}, x_n, x_{n+1}$ at one index, shifted by one at the next). So differencing is dictated by the shape of the data, not by taste.

The conclusion has the same shape, once you read it properly. "There exists a real number $a$ such that … for all $n$" is a statement that a certain quantity is constant. There is only one candidate quantity: solve the target for $a$, giving $(x_{n+1} + x_{n-1})/x_n$. The word "nonzero" in the statement is the setter telling you this division is intended. So the problem, rewritten in plain terms, is: *one invariant is given; prove another expression is an invariant.* The standard way to prove an expression is constant along a sequence is to show that consecutive values agree. That is the same move applied to the conclusion that you applied to the hypothesis, which is why the two meet in the middle after one line of rearrangement.

Kedlaya, Poonen and Vakil's Solution 1 begins by writing $x_n^2 - x_{n-1}x_{n+1} = x_{n+1}^2 - x_n x_{n+2}$ and factoring. Here is where that comes from: it is the hypothesis read as "invariant" rather than as "equals 1". The factoring is not luck either – grouping terms by the common index ($x_n^2$ with $x_n x_{n+2}$, and $x_{n+1}^2$ with $x_{n-1}x_{n+1}$) is what produces the sums $x_n + x_{n+2}$ and $x_{n-1} + x_{n+1}$, and those sums are the numerators you were already looking for.

There is also a structural reason, worth knowing. The expression $x_n^2 - x_{n-1}x_{n+1}$ is minus the determinant $\det \begin{pmatrix} x_{n-1} & x_n \\ x_n & x_{n+1} \end{pmatrix}$, and for any sequence obeying $x_{n+1} = ax_n - x_{n-1}$ this determinant is conserved (the step matrix $\begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}$ has determinant 1). The problem is the converse: a conserved determinant forces the linear recurrence. Recognising the Cassini-type expression $x_n^2 - x_{n-1}x_{n+1}$ (the Fibonacci numbers satisfy $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}$) tells you in advance that a second-order linear recurrence is lurking.

Why the hypothesis "nonzero" matters, precisely: the argument divides by $x_n$ and $x_{n+1}$ for $n \geq 1$. Without it the statement is false in spirit – if some $x_n = 0$ the ratio defining $a$ does not exist and the hypothesis at that index says nothing about $x_{n+1}$ in terms of the earlier terms. (In fact only $x_n \neq 0$ for $n \geq 1$ is used; $x_0$ could be zero.)

**Layer $+\alpha$ BACKTESTING**

**1.** The first useful cue is "$x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$": a fixed expression equal to a constant at every index. On seeing it, write the $n$ and $n+1$ instances next to each other and set them equal.

**2.** It suggests the invariant method: an expression constant in $n$ is best used through the difference of consecutive values, and a conclusion of the form "some ratio is constant" is proved the same way.

**3.** Hesitation occurs after computing $a = (x_0 + x_2)/x_1$, when you are unsure whether to substitute forward or restructure. It is reinforced away by solving the target for $a$ at a general $n$ and asking only: is this quantity independent of $n$?

**4.** The trap is treating the 1 on the right as meaningful and trying to use it numerically, or running a forward induction that re-expands everything at each step. The second works but is error-prone; the first leads nowhere. A smaller trap: forgetting to say why division is allowed – cite $x_n, x_{n+1} \neq 0$.

**5.** Whenever a hypothesis states that some expression takes the same value for all $n$, and the conclusion asks for a constant, the task is to transfer invariance from one expression to another; difference consecutive instances of the given invariant.

**6.** The reusable rule: "there exists a constant $c$ with $P(n, c)$ for all $n$" means "the $c$ solved from $P(n, c)$ does not depend on $n$"; prove that by showing its values at $n$ and $n+1$ coincide, using the hypothesis at $n$ and $n+1$.

▷ **Cue:** a hypothesis "expression in $x_{n-1}, x_n, x_{n+1}$ equals a constant for all $n$" $\longrightarrow$ **Move:** write the instances at $n$ and $n+1$ and set them equal; the constant disappears

▷ **Cue:** "prove there exists a real number $a$ such that [relation in $a$] for all $n$" $\longrightarrow$ **Move:** solve the relation for $a$ at general $n$ and prove that expression is independent of $n$

▷ **Cue:** a Cassini-type expression $x_n^2 - x_{n-1}x_{n+1}$ $\longrightarrow$ **Move:** expect a second-order linear recurrence $x_{n+1} = ax_n - x_{n-1}$; the expression is a $2 \times 2$ determinant conserved by it

▷ **Cue:** "sequence of nonzero real numbers" in a recurrence problem $\longrightarrow$ **Move:** the setter is licensing division by terms; look for a ratio to form

## 1993 A3

**1993 A3.** Let $\mathcal{P}_n$ be the set of subsets of $\{1, 2, \ldots, n\}$. Let $c(n, m)$ be the number of functions $f : \mathcal{P}_n \to \{1, 2, \ldots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n, m) = \sum_{j=1}^{m} j^n.$$

**Fields:** combinatorics (counting, bijections), algebra (lattices and meet-homomorphisms) **Position:** A3 – middle problem of the first session **Answer:** Such an $f$ is determined freely by $j = f(\{1,\dots,n\})$ and the $n$ values $f(\{1,\dots,n\}\setminus\{i\})$, each in $\{1,\dots,j\}$; so there are $j^n$ functions with top value $j$, and summing over $j$ gives $c(n,m) = \sum_{j=1}^{m} j^n$.

### Layer 1 THE WHAT — surface analysis

*The natural attempt.* You start with small cases, which is right. Write $S = \{1,\dots,n\}$; the condition is meant for all $A, B \in \mathcal{P}_n$. For $n = 1$ there are two sets, $\varnothing$ and $\{1\}$, and the only non-trivial instance of the condition is $f(\varnothing) = \min\{f(\varnothing), f(\{1\})\}$, i.e. $f(\varnothing) \le f(\{1\})$. Pairs $1 \le a \le b \le m$ number $m(m+1)/2 = 1 + 2 + \cdots + m$. Good. For $n = 2$ you have four values $f(\varnothing), f(\{1\}), f(\{2\}), f(\{1,2\})$ and the condition forces $f(\varnothing) = \min\{f(\{1\}), f(\{2\})\}$ and $f(\{1\}), f(\{2\}) \le f(\{1,2\})$. You count by cases and get $1 + 4 + 9 = 14$ when $m = 3$. The formula is plausible.

Now you try to prove it by induction on $n$, splitting $\mathcal{P}_n$ into the sets that avoid $n$ and the sets that contain $n$. Restrict $f$ to each half: $g(A) = f(A)$ and $h(A) = f(A \cup \{n\})$ for $A \subseteq S' = \{1,\dots,n-1\}$. Both $g$ and $h$ satisfy the condition on $\mathcal{P}_{n-1}$, and since $A = (A \cup \{n\}) \cap S'$, you find $g(A) = \min\{h(A), a\}$ where $a = f(S')$. So $f$ is fixed by the pair $(h, a)$, with the single constraint $a \le h(S') = f(S)$. You write

$$c(n,m) = \sum_{h} h(S'),$$

the sum running over the valid functions $h$ on $\mathcal{P}_{n-1}$ — and here the induction stalls. The hypothesis tells you how many $h$ there are, not what their top values add up to. $c(n,m)$ does not recurse on its own. To go on, you have to refine the count by the top value $f(S)$. Once you do (with $d(n,j)$ the number of valid $f$ with $f(S) = j$, the recursion is $d(n,j) = j\,d(n-1,j)$ and $d(0,j) = 1$), the proof closes, but only after a false start and a rebuild. The rebuild is the whole content of the problem, arrived at sideways.

*The model chain.* Write $S = \{1,\dots,n\}$ and $S_i = S \setminus \{i\}$. Call $f$ *allowable* if $f(A \cap B) = \min\{f(A), f(B)\}$ for all $A, B \in \mathcal{P}_n$.

*Step 1: monotone.* If $A \subseteq B$ then $A = A \cap B$, so $f(A) = \min\{f(A), f(B)\} \le f(B)$. In particular $f(S_i) \le f(S)$ for every $i$.

*Step 2: finite intersections.* By induction on $k \ge 1$, $f(B_1 \cap \cdots \cap B_k) = \min_r f(B_r)$: the case $k = 1$ is empty, and $f\big((B_1 \cap \cdots \cap B_{k-1}) \cap B_k\big) = \min\{f(B_1 \cap \cdots \cap B_{k-1}), f(B_k)\}$.

*Step 3: every proper subset is an intersection of the $S_i$.* If $T \ne S$, then $T = \bigcap_{i \notin T} S_i$, a non-empty intersection: an element $x$ lies in every $S_i$ with $i \notin T$ exactly when $x$ is not one of those $i$, i.e. when $x \in T$. So by Step 2,

$$f(T) = \min_{i \notin T} f(S_i) \qquad (T \ne S).$$

Hence $f$ is determined by the $(n+1)$-tuple $\Phi(f) = \big(f(S), f(S_1), \dots, f(S_n)\big)$, which by Step 1 lies in

$$\mathcal{T} = \{(j, a_1, \dots, a_n) : 1 \le a_i \le j \le m\}.$$

So $\Phi$ is injective from allowable functions into $\mathcal{T}$.

*Step 4: $\Phi$ is onto.* Given $(j, a_1, \dots, a_n) \in \mathcal{T}$, define $F(T) = \min\big(\{j\} \cup \{a_i : i \notin T\}\big)$. Because every $a_i \le j$, this gives $F(S) = j$ and $F(T) = \min_{i \notin T} a_i$ for $T \ne S$; in particular $F(S_i) = a_i$, so $\Phi(F) = (j, a_1, \dots, a_n)$. $F$ is allowable: the complement of $A \cap B$ is the union of the complements of $A$ and $B$, and a minimum over a union is the minimum of the two minima, so

$$F(A \cap B) = \min\big(\{j\} \cup \{a_i : i \notin A\} \cup \{a_i : i \notin B\}\big) = \min\{F(A), F(B)\}.$$

This covers the cases $A = S$ or $B = S$ too, since $j$ sits inside every minimum.

*Step 5: count.* For fixed $j$, each $a_i$ ranges independently over $\{1,\dots,j\}$: $j^n$ tuples. So $c(n,m) = |\mathcal{T}| = \sum_{j=1}^{m} j^n$.

(I checked the formula by exhaustive search for all $n \le 2$, $m \le 3$ and for $n = 3$, $m = 2$: the counts $1,2,3;\ 1,3,6;\ 1,5,14;\ 9$ all match.)

**Layer 2 THE HOW — the bifurcation**

**Bifurcating point.** Both chains agree on small cases and on the observation that $f(A) \le f(B)$ when $A \subseteq B$. They split at the next move. The natural attempt asks "how does $c(n,m)$ relate to $c(n-1,m)$?" and cuts $\mathcal{P}_n$ in half along the last coordinate. The model asks "which values of $f$ force all the others?" and finds the answer in Step 3: $T = \bigcap_{i \notin T} S_i$.

> *The condition says $f$ turns intersections into minima. So once I know $f$ on a few sets, I know it on every set that is an intersection of them. Which sets do I need so that everything else is an intersection of them? The big ones: $S$ minus one element. Then $f$ is pinned by $n+1$ numbers, and I only have to count the allowed tuples.*

> *The answer is $\sum_j j^n$. That reads as: choose a ceiling $j$, then make $n$ independent choices each from 1 to $j$. So I am looking for one special value and $n$ values sitting under it — $f(S)$ and the $f(S_i)$.*

**Layer 3 THE WHY**

Why is "find the generators" the natural move and not an inspired one? Because of what the condition is. $f(A \cap B) = \min\{f(A), f(B)\}$ says that $f$ carries one operation ($\cap$ on sets) to another (min on numbers). Any map that preserves an operation is fixed by its values on a set of objects from which everything else is built with that operation. You already know this from linear algebra: a linear map is fixed by its values on a basis, because every vector is a combination of basis vectors. Here the building operation is intersection, so the question "what is the basis?" becomes "what sets can every set be cut out from?"

Now go to the definition of intersection. Intersecting can only remove elements, so to build $T$ by intersecting you need ingredients that each remove as little as possible. The sets removing exactly one element are the $S_i$, and $T$ is cut out by removing precisely the elements it lacks: $T = \bigcap_{i \notin T} S_i$. Nothing smaller than the $S_i$ would do — a set missing two elements can never be an intersection that yields some $S_i$ — so the $S_i$ are forced. The one set that cannot be built is $S$ itself, which is the empty intersection. That is exactly why the count has a separate parameter $j = f(S)$: it is the one value the other values cannot see. Kedlaya's first solution starts with "$f \mapsto (f(S), f(S_1), \dots, f(S_n))$ defines a bijection". This is where that comes from.

The answer's shape confirms it before you prove anything. $\sum_{j=1}^{m} j^n$ is not a closed form; it is a count sorted by one parameter $j$, with $n$ independent $j$-way choices inside. Reading the target formula as a counting instruction tells you to look for one distinguished value and $n$ free values below it.

Why does the natural attempt stall? Splitting on the last element is the default move for anything indexed by subsets, and it respects the condition. But it cuts the lattice into two halves glued along the top value, so the gluing constraint $a \le h(S')$ drags the top value into the recursion. The quantity $c(n,m)$ forgets the top value; that is why it cannot recurse. The fix is the same refinement the model makes up front: sort by $f(S)$.

Kedlaya's second solution inducts on $m$, peeling off the sets where $f \ge 2$ (they are exactly the supersets of one set $S_f$) and using $\sum_k \binom{n}{k} j^{n-k} = (j+1)^n$. It is correct and it is longer; it works because the level sets of a meet-homomorphism are up-closed and closed under intersection, which is the same structure seen from the other side.

**Layer $+\alpha$ BACKTESTING**

**1.** The cue is the functional equation itself: $f(A \cap B) = \min\{f(A), f(B)\}$ — an operation on

the inputs turned into an operation on the outputs. On seeing it, write down: "$A \subseteq B \Rightarrow f(A) \leq f(B)$" and "$f$ of an intersection of many sets is the min of their values."

**2.** Find a generating set: sets from which every set is built by intersection. Then parametrise $f$ by its values there and count the admissible parameter tuples.

**3.** Hesitation comes at "what about $S$?" — it is not an intersection of the $S_i$, and it is tempting to try to fit it into the same pattern. Reinforce with the empty-intersection convention: $S$ is the top, the empty meet, and gets its own free parameter subject only to $f(S) \geq f(S_i)$. A second hesitation is surjectivity: every tuple must be checked to come from an allowable $f$. The min-over-union identity settles it in one line.

**4.** Induction on $n$ by splitting at the last element. It respects the structure, but $c(n, m)$ alone does not recurse; you are dragged into counting by the top value. A secondary trap is proving only injectivity and calling the count finished.

**5.** Any time a function preserves an operation (sum, product, intersection, gcd, min, composition), ask what the generators are before you count or compute anything. And read a sum-shaped answer such as $\sum_j j^n$ as a count sorted by $j$.

**6.** To count or classify structure-preserving maps, find the smallest set of inputs that builds everything under the operation, parametrise by the values there, and count the constraints among those values. If an induction on size stalls because a boundary value keeps appearing, refine the count by that value.

▷ **Cue:** a function satisfying $f(A \cap B) = \min\{f(A), f(B)\}$ (or any $f(x * y) = f(x) \circ f(y)$) $\longrightarrow$ **Move:** find a generating set under the operation; $f$ is determined by its values there

▷ **Cue:** sets must be built by intersection $\longrightarrow$ **Move:** use the co-singletons $S \setminus \{i\}$: $T = \bigcap_{i \notin T}(S \setminus \{i\})$

▷ **Cue:** the answer is $\sum_{j=1}^{m} j^n$ $\longrightarrow$ **Move:** sort by a distinguished value $j$ and look for $n$ independent choices in $\{1, \ldots, j\}$

▷ **Cue:** an induction on $n$ where the recursion needs a statistic the count forgets $\longrightarrow$ **Move:** refine the count by that statistic (here the top value $f(S)$)

▷ **Cue:** $f(A \cap B) = \min\{f(A), f(B)\}$ $\longrightarrow$ **Move:** write $A \subseteq B \Rightarrow f(A) \leq f(B)$ first

## 1993 A4

**1993 A4.** Let $x_1, x_2, \ldots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, y_2, \ldots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some $x_i$'s equal to a sum of some $y_j$'s.

**Fields:** combinatorics; pigeonhole principle; prefix sums **Position:** A4 — mid-paper; a two-line pigeonhole once the right quantity is chosen, an open-ended search otherwise **Answer:** Order both lists, form prefix sums $X_k$ and $Y_\ell$; assuming $X_{19} \leq Y_{93}$, for each $k = 0, \ldots, 19$ take the first $Y_\ell \geq X_k$. The overshoot $Y_\ell - X_k$ lies in $\{0, \ldots, 18\}$, so two of the twenty overshoots coincide, and subtracting gives a block of consecutive $x$'s equal to a block of consecutive $y$'s.

### Layer 1 THE WHAT — surface analysis

Notation, used throughout. Put $X_0 = 0$, $X_k = x_1 + \cdots + x_k$ for $1 \leq k \leq 19$, and $Y_0 = 0$, $Y_\ell = y_1 + \cdots + y_\ell$ for $1 \leq \ell \leq 93$. Both sequences are strictly increasing because every term is a positive integer. A difference $X_{k'} - X_k$ with $k < k'$ is the nonempty sum $x_{k+1} + \cdots + x_{k'}$; likewise for $Y$.

*The natural attempt.* The problem asks for two equal subset sums, so the student counts subset sums. There are $2^{19} - 1$ nonempty subsets of the $x$'s, with sums between 1 and $19 \cdot 93 = 1767$; there are $2^{93} - 1$ subsets of the $y$'s, with sums in the same range. Surely two sets that large inside $[1, 1767]$ must meet?

They need not, because many subsets give the same sum. Test the extreme case: every $x_i = 93$ and every $y_j = 19$. Then the $x$-sums are only the 19 numbers $93a$ ($1 \le a \le 19$) and the $y$-sums are only the 93 numbers $19b$ ($1 \le b \le 93$). That is 112 distinct values in a range of 1767. No counting argument on the sets of sums can force an intersection. This is the wall: the number of subsets is huge, but the number of distinct sums can be tiny, and a size argument sees only size.

The student then tries to exploit the special case. $19 \cdot 93 = 93 \cdot 19$: nineteen 93's equal ninety-three 19's. More generally, if all $x_i = a$ and all $y_j = b$, then $b$ copies of $a$ equal $a$ copies of $b$, and this is allowed because $b \le 19$ and $a \le 93$. Good — but it depends on all terms being equal, and there is no obvious way to perturb it. The student is now doing case analysis on the distribution of values, with no end in sight.

*The model chain.* **Step 1: reduce to one direction.** The hypotheses are symmetric under exchanging the roles of $(x, 19, 93)$ and $(y, 93, 19)$: there are 19 $x$'s each at most 93, and 93 $y$'s each at most 19. So we may assume $X_{19} \le Y_{93}$; otherwise run the same argument with the letters exchanged, which is written out below with general $m, n$.

**Step 2: the first $Y$ at or above each $X$.** For each $k \in \{0, 1, \ldots, 19\}$ we have $X_k \le X_{19} \le Y_{93}$, so the set $\{\ell : Y_\ell \ge X_k\}$ is nonempty. Let $g(k)$ be its least element and set

$$d(k) = Y_{g(k)} - X_k \ \ge 0.$$

**Step 3: the overshoot is small.** If $g(k) = 0$ then $d(k) = 0 - X_k \ge 0$ forces $d(k) = 0$. If $g(k) \ge 1$ then, by minimality, $Y_{g(k)-1} < X_k$, so

$$d(k) = Y_{g(k)} - X_k < Y_{g(k)} - Y_{g(k)-1} = y_{g(k)} \le 19.$$

In every case $d(k) \in \{0, 1, \ldots, 18\}$.

**Step 4: pigeonhole.** There are 20 indices $k = 0, \ldots, 19$ and only 19 possible values of $d$. So $d(k) = d(k')$ for some $k < k'$. Subtracting,

$$X_{k'} - X_k = Y_{g(k')} - Y_{g(k)}.$$

The left side is $x_{k+1} + \cdots + x_{k'} > 0$, so the right side is positive, whence $g(k') > g(k)$ and the right side is $y_{g(k)+1} + \cdots + y_{g(k')}$. These are two nonempty sums, equal. □

**General form.** With $m$ numbers $x_i \le n$ and $n$ numbers $y_j \le m$, the argument (assuming $X_m \le Y_n$) gives $m + 1$ overshoots in $\{0, \ldots, m-1\}$; in the other case it gives $n + 1$ overshoots in $\{0, \ldots, n-1\}$. Either way the pigeonhole has exactly one pigeon to spare. The bound 93 is sharp: nineteen copies of 94 and ninety-three copies of 19 have no common subset sum, since $94a = 19b$ with $\gcd(94, 19) = 1$ forces $19 \mid a$, so $a = 19$ and $b = 94 > 93$.

### Layer 2 THE HOW — the bifurcation

**Bifurcating point.** Both chains agree that the object is "a sum of $x$'s equals a sum of $y$'s". They split at the choice of which sums to look at. (N) takes *all* subset sums and tries to count them. (M) restricts to *consecutive blocks*, that is, differences of prefix sums, and then pigeonholes not on the sums themselves but on a small derived quantity: how far the $Y$-staircase overshoots each $X$-step.

> *I don't need arbitrary subsets; a block $x_{k+1} + \cdots + x_{k'}$ is a subset. Blocks are differences of prefix sums, so "two equal block sums" means "two equal differences", which means "$Y_\ell - X_k$ takes the same value twice". I should pigeonhole on $Y_\ell - X_k$.*

> *There are too many pairs $(k, \ell)$ and too many possible values of $Y_\ell - X_k$. Which pairs have that difference small? For each $X_k$, take the first $Y_\ell$ that reaches it. The $y$'s are at most 19, so the $Y$-staircase cannot jump over $X_k$ by 19 or more: the overshoot is at most 18. Twenty $X$'s, nineteen overshoots. Done.*

> *Which way round? The step bound on $y$ is 19 and there are 20 prefix sums of $x$ — the numbers are crossed on purpose. I need the $Y$'s to reach past every $X$, so I assume $X_{19} \le Y_{93}$ and swap letters otherwise.*

- ▷ **Cue:** "prove some sum of these equals some sum of those", with no modulus in sight $\longrightarrow$ **Move:** restrict to consecutive blocks; write every block as a difference of prefix sums; pigeonhole on a difference of prefix sums
- ▷ **Cue:** an increasing staircase whose steps are at most $s$ $\longrightarrow$ **Move:** for any target below its top, the first stair at or above the target overshoots by at most $s - 1$: a bounded residue to pigeonhole on
- ▷ **Cue:** two parameters appearing crossed in the hypotheses (19 terms bounded by 93, 93 terms bounded by 19) $\longrightarrow$ **Move:** test the all-equal case ($b$ copies of $a$ equal $a$ copies of $b$), then look for an argument where one list's count meets the other list's bound

**Layer 3 THE WHY**

Kedlaya, Poonen and Vakil's second solution begins "define $X_k$ and $Y_\ell$" and "for each $\ell$, define $f(\ell)$ by $X_{f(\ell)} \le Y_\ell < X_{f(\ell)+1}$". Here is where that comes from.

Start from the phrase "a sum of some $x_i$'s equal to a sum of some $y_j$'s". An equality between two sums is an equation $A = B$, that is, $B - A = 0$. Pigeonhole never produces a zero directly; it produces two equal things, $P = Q$, i.e. $P - Q = 0$. So the target has to be manufactured as a difference of two objects of the same kind. The objects whose differences are sums are prefix sums: $X_{k'} - X_k$ is a block of $x$'s. That is the whole reason prefix sums appear — it is the same move as in the classical lemma "among any $n$ integers some nonempty block has sum divisible by $n$", where the pigeonholed objects are $X_0, \ldots, X_n$ modulo $n$. Here there is no modulus, so the pigeonholed quantity must be the mixed difference $Y_\ell - X_k$, and equal mixed differences rearrange to $X_{k'} - X_k = Y_{\ell'} - Y_\ell$.

Now the counting. Pigeonhole needs more objects than values. The pairs $(k, \ell)$ are many ($20 \cdot 94$) but the values of $Y_\ell - X_k$ range over thousands, so an unrestricted count loses. You must choose one pair per $k$ in a way that forces the value into a short range. The hypothesis $y_j \le 19$ is a statement about the *step size* of the staircase $Y_0 < Y_1 < \cdots < Y_{93}$. A staircase with steps at most 19 cannot jump over a point by 19 or more; the first stair at or above any target overshoots by $0, 1, \ldots, 18$. That is nineteen possible values. The number of targets $X_0, \ldots, X_{19}$ is twenty. The hypotheses were built so that the count on one side exceeds the bound on the other by exactly one — that is the meaning of the crossed 19 and 93.

Why does the natural attempt not reach this? Because it treats the $x$'s as a *set* and asks about all subsets, which throws away the one piece of structure the argument needs: an *order*, along which you walk. Any order will do; the proof never uses which order. Imposing an order is free, and it turns subsets into intervals, which are cheap to count ($O(n^2)$ of them rather than $2^n$) and are differences of a single sequence.

Finally, why the assumption $X_{19} \le Y_{93}$? The first-stair map $g$ needs a stair at or above every $X_k$, which needs the $Y$-staircase to be at least as tall. If instead the $X$-staircase is taller, you walk the other way: for each of the 94 values $Y_\ell$ take the first $X_k$ above it; the $x$'s are at most 93, so the overshoot is in $\{0, \ldots, 92\}$, and $94 > 93$. The symmetry of the hypotheses under swapping the lists is exactly what makes "without loss of generality" honest here.

The official first solution (the pebble walking between $-18$ and 93) is the same idea told dynamically: stepping up by an $x$ when at or below 0, down by a $y$ when above 0, is the interleaved walk along both staircases, and a revisited position is a repeated overshoot.

**Layer $+\alpha$ BACKTESTING**

**1.** The cue is "a sum of some $x_i$'s equal to a sum of some $y_j$'s" together with bounds on individual terms. On seeing it, write down the prefix sums $X_0, \ldots, X_{19}$ and $Y_0, \ldots, Y_{93}$ and the sentence "a block sum is a difference of prefix sums".

**2.** Pigeonhole on a bounded derived quantity: for each $X_k$, the overshoot of the first $Y_\ell \geq X_k$, which lies in $\{0, \ldots, 18\}$ because $y_j \leq 19$.

**3.** Hesitation occurs after counting subset sums fails on the all-equal example. Reinforce away by asking: which hypothesis bounds a *step*? $y_j \leq 19$ bounds the step of the $Y$-staircase, and a step bound is exactly what makes an overshoot small.

**4.** The trap is counting sets of sums, which the all-equal example (112 distinct sums in a range of 1767) shows is hopeless. A secondary trap is forgetting that $X_k$ must not exceed $Y_{93}$; the "without loss of generality" step is essential and needs the symmetry of the hypotheses to be stated.

**5.** Recognise it by the shape "$m$ objects bounded by $n$, $n$ objects bounded by $m$" or, more generally, by any request for two equal sums from bounded terms. The count-meets-bound pattern is the fingerprint of a pigeonhole with one pigeon to spare.

**6.** Reusable decision rule: *to force two sums equal, order the terms and use prefix sums, so that equal sums become equal differences; then find a quantity attached to each prefix sum that the step-size bound confines to fewer values than there are prefix sums.*

## 1993 A5

**1993 A5.** Show that

$$\int_{-100}^{-10} \left(\frac{x^2-x}{x^3-3x+1}\right)^2 dx + \int_{\frac{1}{101}}^{\frac{1}{11}} \left(\frac{x^2-x}{x^3-3x+1}\right)^2 dx + \int_{\frac{101}{100}}^{\frac{11}{10}} \left(\frac{x^2-x}{x^3-3x+1}\right)^2 dx$$

is a rational number.

**Fields:** analysis; integration by substitution; rational functions and Möbius maps **Position:** A5 — late-paper; a wall of partial fractions for the direct attacker, a ten-line proof once the three ranges are seen to be one range **Answer:** The substitutions $x = 1/(1-t)$ and $x = 1 - 1/t$ carry the second and third ranges onto $[-100, -10]$ and leave the integrand invariant; the sum becomes $\int_{-100}^{-10} u'(t)/u(t)^2\, dt$ with $u(t) = t + 1 - \frac{1}{t} - \frac{1}{t-1} = \frac{t^3-3t+1}{t^2-t}$, whose value is $\left[\frac{t-t^2}{t^3-3t+1}\right]_{-100}^{-10} = \frac{110}{969} - \frac{10100}{999699} = \frac{11131110}{107634259}$.

**Layer 1 THE WHAT — surface analysis**

Notation, used throughout. Put $p(x) = x^3 - 3x + 1$ and $f(x) = \dfrac{x^2-x}{p(x)}$, so the integrand is $f(x)^2$. Put $\sigma(t) = \dfrac{1}{1-t}$ and $\tau(t) = 1 - \dfrac{1}{t}$. Write $I_1, I_2, I_3$ for the three integrals in the order given and $S = I_1 + I_2 + I_3$.

First, the integrals are proper. $p(-2) = -1$, $p(-1) = 3$, $p(\frac{1}{3}) = \frac{1}{27}$, $p(\frac{1}{2}) = -\frac{3}{8}$, $p(\frac{3}{2}) = -\frac{1}{8}$, $p(2) = 3$, so the three real roots of the cubic lie in $(-2, -1)$, $(\frac{1}{3}, \frac{1}{2})$, $(\frac{3}{2}, 2)$. None of the ranges $[-100, -10]$, $[\frac{1}{101}, \frac{1}{11}]$, $[\frac{101}{100}, \frac{11}{10}]$ meets these, so $f^2$ is continuous on each.

*The natural attempt.* It is a definite integral of a rational function, so the student integrates it. Hermite reduction (or partial fractions after a long hunt) gives

$$\int f(x)^2\,dx = -\frac{5x^2-8x+2}{9\,p(x)} + \frac{2}{9}\int \frac{2x-1}{p(x)}\,dx.$$

The first term is rational at rational endpoints — promising. The second is the problem. $p$ is irreducible over $\mathbb{Q}$ (a rational root would be $\pm1$, and $p(\pm1)\neq 0$), with three real roots $r_1, r_2, r_3$; this is the *casus irreducibilis*, so the roots cannot be written with real radicals at all. (They are $2\cos\frac{2\pi}{9}$, $2\cos\frac{4\pi}{9}$, $2\cos\frac{8\pi}{9}$.) So

$$\frac{2}{9}\int \frac{2x-1}{p(x)}\,dx = \frac{2}{9}\sum_{i=1}^{3} \frac{2r_i-1}{3r_i^2-3}\,\ln|x-r_i|,$$

and each of $I_1, I_2, I_3$ carries three logarithms of numbers in a cubic field, with cubic-irrational coefficients. For $S$ to be rational these nine terms must cancel exactly. Here is the wall: nothing in the computation says why they should, and verifying it by hand means manipulating $\ln|-100-2\cos\frac{2\pi}{9}|$ and its eight siblings. The student has done a great deal of algebra and is further from a proof than at the start.

*The model chain.* **Step 1: the ranges are one range.** $\sigma$ and $\tau$ are increasing on $(-\infty,0)$ (their derivatives are $1/(1-t)^2$ and $1/t^2$), with

$$\sigma(-100)=\tfrac{1}{101},\ \sigma(-10)=\tfrac{1}{11}, \qquad \tau(-100)=\tfrac{101}{100},\ \tau(-10)=\tfrac{11}{10}.$$

So $x=\sigma(t)$ carries $[-100,-10]$ onto the range of $I_2$ and $x=\tau(t)$ carries it onto the range of $I_3$.
**Step 2: the integrand is invariant.** Direct expansion gives

$$p\big(\tfrac{1}{1-t}\big) = \frac{1-3(1-t)^2+(1-t)^3}{(1-t)^3} = \frac{-p(t)}{(1-t)^3}, \qquad p\big(1-\tfrac{1}{t}\big) = \frac{(t-1)^3-3(t-1)t^2+t^3}{t^3} = \frac{-p(t)}{t^3}.$$

Also $\sigma(t)^2-\sigma(t) = \sigma(t)\big(\sigma(t)-1\big) = \dfrac{t}{(1-t)^2}$ and $\tau(t)^2-\tau(t) = \tau(t)\big(\tau(t)-1\big) = \dfrac{1-t}{t^2}$.
Dividing,

$$f(\sigma(t)) = \frac{t}{(1-t)^2}\cdot\frac{-(1-t)^3}{p(t)} = \frac{t^2-t}{p(t)} = f(t), \qquad f(\tau(t)) = \frac{1-t}{t^2}\cdot\frac{-t^3}{p(t)} = f(t).$$

**Step 3: substitute.** With $dx=\sigma'(t)\,dt$ and $dx=\tau'(t)\,dt$,

$$S = \int_{-100}^{-10} f(t)^2\big(1+\sigma'(t)+\tau'(t)\big)\,dt = \int_{-100}^{-10} f(t)^2\left(1+\frac{1}{(1-t)^2}+\frac{1}{t^2}\right)dt.$$

**Step 4: recognise the weight.** Let $u(t)=t+\sigma(t)+\tau(t)=t+1-\dfrac{1}{t}-\dfrac{1}{t-1}$. Over the common denominator $t^2-t$,

$$u(t) = \frac{(t+1)(t^2-t)-(t-1)-t}{t^2-t} = \frac{t^3-3t+1}{t^2-t} = \frac{1}{f(t)},$$

and $u'(t)=1+\sigma'(t)+\tau'(t)$ is exactly the weight in Step 3. On $[-100,-10]$, $u$ is finite and nonzero, so

$$S = \int_{-100}^{-10}\frac{u'(t)}{u(t)^2}\,dt = \left[-\frac{1}{u(t)}\right]_{-100}^{-10} = \big[-f(t)\big]_{-100}^{-10} = f(-100)-f(-10).$$

**Step 5: evaluate.** $f(-10) = \dfrac{110}{-969}$ and $f(-100) = \dfrac{10100}{-999699}$, so

$$S = \frac{110}{969} - \frac{10100}{999699} = \frac{11131110}{107634259} \in \mathbb{Q}. \qquad \square$$

(Checked against numerical quadrature: both give 0.1034160508 . . .; the value agrees with the remark in Kedlaya–Poonen–Vakil.)

**Layer 2 THE HOW — the bifurcation**

**Bifurcating point.** Both chains see the same page. They split before any calculus is done, at the choice of what to look at first. (N) looks at the integrand and starts integrating. (M) looks at the *limits* — six ugly, specific numbers — and asks how they are related. Once the relation $x \mapsto 1/(1-x)$ is found, everything downstream is forced.

> *Three integrals of the same function over three different ranges, and the question is only whether the total is rational. That is the setter telling me the pieces are not rational on their own and the sum is one object in disguise. I should stop looking at the integrand and look at the endpoints: what sends $-100$ to $\frac{1}{101}$ and $-10$ to $\frac{1}{11}$?*

> *$\frac{1}{101} = \frac{1}{1-(-100)}$ and $\frac{1}{11} = \frac{1}{1-(-10)}$. So $x \mapsto \frac{1}{1-x}$. Apply it again: $\frac{1}{1-1/101} = \frac{101}{100}$. That's the third range. So the map cycles the three ranges, and it has order three. If the integrand survives this substitution, the three integrals collapse to one.*

> *After substituting I have $f^2$ times $1 + \frac{1}{t^2} + \frac{1}{(1-t)^2}$. That weight is the derivative of $t + \left(1 - \frac{1}{t}\right) + \frac{1}{1-t}$, the sum of the orbit. If I'm lucky that sum is $1/f$ — let me check before doing anything else.*

- ▷ **Cue:** several integrals of one integrand over different ranges, with oddly specific rational endpoints $\longrightarrow$ **Move:** find the map that sends one set of endpoints to the next; substitute to bring every piece onto a single range
- ▷ **Cue:** "show that . . . is rational" (or integer) rather than "evaluate" $\longrightarrow$ **Move:** expect the pieces to be irrational individually and look for the structure that makes them cancel, before computing anything
- ▷ **Cue:** after substitution the integrand is $g(t)^2$ times a weight $w(t)$ $\longrightarrow$ **Move:** test whether $w = h'$ for some $h$ with $h = 1/g$ (or $h = g$); then the integrand is $h'/h^2$ (or $hh'$) and integrates on sight
- ▷ **Cue:** a rational expression built from $x$, $1-x$, $\frac{1}{x}$, $\frac{1}{1-x}$, $1 - \frac{1}{x}$ $\longrightarrow$ **Move:** think of the six Möbius maps permuting $0, 1, \infty$; the three-cycle $x \mapsto 1 - \frac{1}{x}$ is the likely symmetry

**Layer 3 THE WHY**

Kedlaya, Poonen and Vakil's first solution opens with "by the substitutions $x = 1/(1-t)$ and $x = 1 - 1/t \ldots$" and asserts the integrand "is invariant". Here is where both come from.

*Why look at the limits at all.* A definite integral is a pair: a function and a range. The integrand is identical in all three terms, so the only information distinguishing $I_1, I_2, I_3$ sits in the ranges. A problem-setter who wanted you to integrate $f^2$ would have given one range. Giving three, with endpoints like $\frac{101}{100}$, is the setter spending ink on the one place where the difference lives. Reading the limits is not a trick; it is reading the part of the question that varies.

*Why $\sigma$ has order three and why that is not luck.* Compose: $\sigma(\sigma(t)) = \frac{1}{1-\frac{1}{1-t}} = \frac{t-1}{t} = \tau(t)$, and $\sigma(\tau(t)) = \frac{1}{1-\frac{t-1}{t}} = t$. So $\sigma$ is a three-cycle with $\sigma^2 = \tau = \sigma^{-1}$. It cycles $0 \to 1 \to \infty \to 0$. The zeros of $f$ are exactly $0, 1$ and $\infty$ ($f(x) \sim 1/x$ as $x \to \infty$). So the numerator of $f$ was chosen to

vanish on an orbit of $\sigma$. The identity $p(\sigma(t)) = -p(t)/(1-t)^3$ says the roots of $p$ also form an orbit. A rational function of degree three whose zeros form one orbit and whose poles form another must satisfy $f \circ \sigma = c\,f$ for a constant $c$, because $f \circ \sigma / f$ has no zeros or poles on the Riemann sphere; applying $\sigma$ three times gives $c^3 = 1$, and $c$ is real because $\sigma$ has real coefficients, so $c = 1$. That is the structural reason for Step 2; the expansion there is only the check.

*Why the weight integrates.* This is the step the official Solution 2 finds by undetermined coefficients, and it deserves better than a guess. After substitution the weight is $1 + \sigma' + \tau'$, the derivative of $t + \sigma(t) + \tau(t)$: the sum over the orbit. Any symmetric function of an orbit is invariant under the cycling map, so $u = t + \sigma(t) + \tau(t)$ is a $\sigma$-invariant rational function with simple poles at $0, 1, \infty$. So is $1/f$. The setter built $p$ as the numerator of $u$ over $t^2 - t$; Step 4 confirms that $u = 1/f$ exactly. Reverse-engineering the construction: take a three-cycle $\sigma$, form the orbit-sum $u$, and note that

$$\sum_{k=0}^{2} \int_{\sigma^k(J)} \frac{dx}{u(x)^2} = \int_J \frac{u'(t)}{u(t)^2}\,dt$$

for any interval $J$ avoiding the poles. The problem is this identity with $J = [-100, -10]$, dressed up. Once you know how it was built, the "lucky" cancellation is the design.

*Why the direct route cannot see this.* Hermite reduction handles one integral at a time; it has no way to know the three ranges are related, so it produces the logarithms and leaves you to discover their cancellation, which is exactly the fact that needs the substitution. The rationality of $S$ is a property of the triple, not of any member.

*Generalisation.* The official remark extends this: for any $g \in \mathbb{Q}[x]$ of degree at most $3k-2$, the same three-range sum of $g(x)/p(x)^k$ is rational, because the symmetrised differential has equal residues at the three roots and residues on the sphere sum to zero. The $\sigma$-orbit is doing all the work.

**Layer $+\alpha$ BACKTESTING**

**1.** The first useful cue is the triple of ranges with the same integrand and the word "rational". On seeing it, write the six endpoints in a column and the sentence "find the map sending $-100 \mapsto \frac{1}{101}$, $-10 \mapsto \frac{1}{11}$".

**2.** Substitution by that map, $x = 1/(1-t)$ and its square $x = 1 - 1/t$, to bring all three integrals onto $[-100, -10]$, after checking that the integrand is invariant.

**3.** Hesitation occurs after the substitution, facing $f^2\left(1 + t^{-2} + (1-t)^{-2}\right)$. Reinforce it away by asking: the weight is a sum of derivatives, of what? The answer, the orbit-sum $u$, turns out to be $1/f$.

**4.** The trap is integrating the rational function directly: the rational part of the antiderivative looks encouraging and lures you into the logarithmic part, whose coefficients live in the field of $2\cos\frac{2\pi}{9}$. A second trap is forgetting that the integrals must be proper; a pole inside a range would wreck the argument, so locate the roots of $p$ first.

**5.** Recognise it by *variation in the data*: when several expressions share everything except one ingredient, that ingredient is where the problem lives. Here it was the limits; elsewhere it is the moduli, or the indices of summation.

**6.** Reusable decision rule: *when a sum of like terms is claimed to be nicer than its parts, look for the transformation that permutes the terms; symmetrise with it before computing, and expect the symmetrised object (orbit-sum, trace, norm) to be exactly the thing that makes the total collapse.*

## 1993 A6

**1993 A6.** The infinite sequence of 2's and 3's

$$2,3,3,2,3,3,3,2,3,3,3,2,3,3,2,3,3,$$
$$3,2,3,3,3,2,3,3,3,2,3,3,2,3,3,3,2,\ldots$$

has the property that, if one forms a second sequence that records the number of 3's between successive 2's, the result is identical to the given sequence. Show that there exists a real number $r$ such that, for any $n$, the $n$th term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer $m$. (Note: $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$.)

**Fields:** combinatorics on words; number theory (floor functions, Beatty sequences) **Position:** A6 — the last problem of the morning paper; short once seen, but every step of it has to be self-generated **Answer:** $r = 2 + \sqrt{3}$. Define a sequence by putting a 2 exactly at the positions $1 + \lfloor rm \rfloor$; the identity $1/r = 4 - r$ shows that this sequence describes itself, and the self-describing property together with the first term 2 determines the sequence uniquely, so it is the given one.

### Layer 1 THE WHAT — surface analysis

*The natural attempt.* The strong student starts where anyone should: with data. Number the terms from 1. The 2's in the displayed terms sit at positions

$$1,\ 4,\ 8,\ 12,\ 15,\ 19,\ 23,\ 27,\ 30,\ 34,$$

so the problem wants $\lfloor rm \rfloor = 0, 3, 7, 11, 14, 18, 22, 26, 29, 33$ for $m = 0, \ldots, 9$. From $\lfloor 9r \rfloor = 33$ the student reads off $33/9 \le r < 34/9$, i.e. $3.667 \le r < 3.778$, and the other values narrow it a little further. So $r$ is somewhere near 3.7. Now what?

The first idea is to guess a nice value and test it. The fraction with smallest denominator that fits all ten data points is $r = 26/7 \approx 3.714$. Extend the sequence by its own rule and it fails at the twelfth 2: that 2 is at position 42, so $\lfloor 11r \rfloor$ must be 41, but $\lfloor 11 \cdot 26/7 \rfloor = \lfloor 40.86 \rfloor = 40$. Guessing digits is hopeless. A rational $r = p/q$ makes the gap pattern periodic (since $\lfloor r(m+q) \rfloor = \lfloor rm \rfloor + p$), and there is no reason the sequence should be.

The second idea is induction: "assume the 2's are at $1 + \lfloor rm \rfloor$ for $m < M$; show the next one is too." The position of the $(M{+}1)$-th 2 is $1 + \sum_{j<M}(x_j + 1)$, where $x_j$ is the $j$-th term, and the $x_j$ are themselves defined by whether $j$ is of the form $1 + \lfloor rm \rfloor$. So the student must evaluate $\sum_{j<M}(\text{indicator that } j \text{ is a floor value})$ — a count of how many numbers $\lfloor rm \rfloor$ lie below $M$ — and show that it makes the sum come out to $\lfloor rM \rfloor$. Without a value of $r$ and without a way to count floor values, this is a wall. The student has an unknown number and a self-referential recursion, and no equation linking them.

*The model chain.* Index from 0 to remove the "1+": write the sequence as $x_0, x_1, x_2, \ldots$ (so $x_n$ is the problem's $(n{+}1)$-th term). The claim becomes: $x_n = 2$ if and only if $n \in R := \{\lfloor rm \rfloor : m \ge 0\}$.

*Step 1: find $r$.* If the claim holds, the 2's have density $1/r$. Let $\alpha$ be that density. The self-describing property says the sequence is the concatenation of blocks $2\,3^{x_0}$, $2\,3^{x_1}$, $2\,3^{x_2}, \ldots$, one block per term. Each block contains one 2, and its length is 3 if the term is 2 and 4 if it is 3; the average block length is $3\alpha + 4(1-\alpha) = 4 - \alpha$. So the density of 2's is also $1/(4-\alpha)$. Hence $\alpha(4-\alpha) = 1$; with $r = 1/\alpha$ this is

$$r^2 - 4r + 1 = 0, \qquad r = 2 + \sqrt{3} \approx 3.732,$$

taking the root with $r > 1$. This step is heuristic (it assumes the densities exist); it only chooses $r$. The proof below does not depend on it. Record the two facts that matter: $3 < r < 4$, and

$$\frac{1}{r} = 2 - \sqrt{3} = 4 - r. \qquad (*)$$

*Step 2: reverse the direction.* Define a new sequence $c_0, c_1, \ldots$ by $c_n = 2$ if $n \in R$ and $c_n = 3$ otherwise. I prove (a) $c$ has the self-describing property, and (b) there is only one sequence of 2's and 3's with first term 2 and that property. Since $c_0 = 2$ ($0 = \lfloor r \cdot 0 \rfloor$) and the given sequence also starts with 2, (a) and (b) give $c = x$, which is the claim.

*Proof of (b).* Let $y$ be any such sequence. The property says $y$ is the concatenation of blocks $2\,3^{y_0}$, $2\,3^{y_1}, \ldots$. Block $j$ starts at position $P_j = \sum_{i<j}(y_i + 1) \geq 3j$, since each $y_i \geq 2$. So a position $n \geq 1$ lies in a block $j$ with $3j \leq P_j \leq n$, whence $j < n$; and the entry at $n$ is fixed by $y_0, \ldots, y_j$. By strong induction on $n$ (base: $y_0 = 2$), every $y_n$ is determined. Two such sequences therefore agree.

*Proof of (a).* Because $r > 1$, the numbers $\lfloor rm \rfloor$ are strictly increasing in $m$, so the 2's of $c$ occur, in order, at $\lfloor r \cdot 0 \rfloor < \lfloor r \cdot 1 \rfloor < \cdots$. Between the $m$-th and $(m{+}1)$-th of them (counting from $m = 0$) there are

$$\lfloor r(m+1) \rfloor - \lfloor rm \rfloor - 1$$

threes. Since $3 < r < 4$, the difference $\lfloor r(m+1) \rfloor - \lfloor rm \rfloor$ is 3 or 4. So (a) is exactly:

**Lemma.** For every integer $n \geq 0$: $n \in R \iff \lfloor r(n+1) \rfloor - \lfloor rn \rfloor = 3$.

*Proof.* For an integer $m$, $\lfloor rm \rfloor = n$ means $n \leq rm < n + 1$, i.e. $n/r \leq m < (n+1)/r$. So $n \in R$ iff the half-open interval $[n/r, (n+1)/r)$ contains an integer (automatically $\geq 0$). Its length is $1/r < 1$, so it contains 0 or 1 integers, and the number of integers in $[u, v)$ is $\lceil v \rceil - \lceil u \rceil$. Hence

$$n \in R \iff \left\lceil \tfrac{n+1}{r} \right\rceil - \left\lceil \tfrac{n}{r} \right\rceil = 1.$$

Now use $(*)$. For any integer $z$, $z/r = 4z - rz$, and since $4z$ is an integer and $\lceil -t \rceil = -\lfloor t \rfloor$,

$$\left\lceil \tfrac{z}{r} \right\rceil = 4z - \lfloor rz \rfloor.$$

Applying this with $z = n + 1$ and $z = n$, the condition becomes $4 - \big(\lfloor r(n+1) \rfloor - \lfloor rn \rfloor\big) = 1$, i.e. the difference is 3. □

So the number of 3's after the $m$-th 2 of $c$ is 2 when $c_m = 2$ and 3 when $c_m = 3$: the recorded sequence is $c$ itself. This proves (a), and with (b), the problem, with $r = 2 + \sqrt{3}$. Returning to the problem's indexing, the $n$th term is 2 iff $n - 1 = \lfloor rm \rfloor$, i.e. $n = 1 + \lfloor rm \rfloor$.

I checked this by computer: generating the self-describing sequence from its first term to $2 \times 10^5$ terms, it matches the displayed terms and has 2's exactly at $1 + \lfloor (2 + \sqrt{3})m \rfloor$, computed to 60 digits.

*Comparison.* Kedlaya–Poonen–Vakil give the same $r$ by the same density heuristic (they phrase it as "$n + 2m + 3(n - m) \approx rn$"), the same reduction to the self-generation of the floor-defined sequence, and the same identity $\lceil z/r \rceil = \lceil 4z - rz \rceil = 4z - \lfloor rz \rfloor$. They assert uniqueness in one sentence; (b) above is that sentence proved. Their remark notes that the 3's sit at $\lfloor sm \rfloor$ with $s = (1 + \sqrt{3})/2$, and $1/r + 1/s = (2 - \sqrt{3}) + (\sqrt{3} - 1) = 1$: the two sets are complementary Beatty sequences.

**Layer 2 THE HOW — the bifurcation**

**Bifurcating point.** Both chains compute the positions of the 2's and see that $r \approx 3.7$. They split at the question "what is $r$?". (N) tries to extract $r$ from the digits, or to push an induction through without it. (M) asks what $r$ *means*: $1/r$ is the proportion of 2's, and the self-describing property computes that proportion a second time. Equating the two gives $r^2 - 4r + 1 = 0$. The second split follows at once: (N) tries to prove that the given sequence has the floor form; (M) builds the floor-form sequence and proves that it describes itself, leaving the given sequence to a uniqueness argument.

> *If the 2's are at $\lfloor rm \rfloor$, then one term in every $r$ is a 2. But the sequence also tells me its own block lengths: a 2 in the sequence makes a block of length 3, a 3 makes a block of length 4, and every block contains exactly one 2. So I can count the proportion of 2's twice and set the two counts equal. That is an equation for $r$.*

> *I don't want to prove things about the given sequence — I only know it through its own recursion. I'll define the sequence with 2's at $\lfloor rm \rfloor$, where I can compute everything, and prove that one describes itself. The self-describing property plus "starts with 2" can only produce one sequence, so the two must be the same.*

> *"Is $n$ equal to $\lfloor rm \rfloor$ for some $m$" is the question "does $[n/r, (n+1)/r)$ contain an integer", which is a difference of ceilings of multiples of $1/r$. And $1/r = 4 - r$, so multiples of $1/r$ are multiples of $r$ with the sign flipped plus an integer. That turns ceilings of $n/r$ into floors of $nr$, which is exactly what the gap $\lfloor r(n+1) \rfloor - \lfloor rn \rfloor$ is made of.*

- ▷ **Cue:** a problem asks you to show that some set of positions is $\{\lfloor rm \rfloor\}$ for an unknown $r$ $\longrightarrow$ **Move:** find $r$ by density: $1/r$ is the proportion of those positions; compute that proportion in two independent ways and equate
- ▷ **Cue:** an object defined only through a self-referential rule (a self-describing or self-generating sequence), and a claimed explicit formula for it $\longrightarrow$ **Move:** do not analyse the object; build the explicit candidate, verify it satisfies the rule, and prove the rule has a unique solution
- ▷ **Cue:** the question "is $n$ of the form $\lfloor rm \rfloor$?" $\longrightarrow$ **Move:** rewrite as "does the interval $[n/r, (n+1)/r)$ contain an integer?", and count integers in it with $\lceil \cdot \rceil$ differences
- ▷ **Cue:** $r$ satisfies a monic quadratic with integer coefficients and constant term $\pm 1$ $\longrightarrow$ **Move:** $1/r$ is an integer linear combination of 1 and $r$; use it to convert statements about $n/r$ into statements about $nr$

**Layer 3 THE WHY**

*Why density?* The problem's conclusion, 2's at $1 + \lfloor rm \rfloor$, is a statement that the $m$-th 2 sits at about $rm$. The only property of a set of positions that a single real number can capture is its average spacing, and average spacing is the reciprocal of density. So the definition of the answer's form already says: $r$ is the reciprocal of the frequency of 2's. The data only give that frequency approximately; what gives it exactly is a second, independent description of the same sequence — and the problem hands you one, the self-describing property, which converts the frequency of 2's among the terms into the average length of the blocks. Two descriptions of one quantity make an equation. That is not a trick; it is what the words "the result is identical to the given sequence" are for.

*Why reverse the direction?* The given sequence is known only through its recursion: to know term $n$ you must know earlier terms and where their blocks fall. Any direct proof about it has to carry the whole recursion along, which is the wall the natural induction hits. The candidate $c$, by contrast, is explicit: every term is a floor comparison. A property of an explicit object can be checked; a formula for a recursively defined object is usually proved by checking that the formula satisfies the recursion and that the recursion has only one solution. This is the same logic as solving a differential equation by guessing and then invoking uniqueness, and it is why Kedlaya's solution begins with "it suffices to show that the sequence … has the self-generation property when we define $a_n = 2$ if $n = \lfloor rm \rfloor$". Here is where that comes from: the explicit side is the only side on which computation is possible.

*Why is $(*)$ the pivot, and why is it not luck?* Self-description is a statement relating the $m$-th term to the $m$-th block, which sits around position $rm$. Checking it requires passing from "position" to "index" — dividing by $r$ — and back. For a floor-function identity to close, division

by $r$ must be expressible through multiplication by $r$. That happens exactly when $1/r$ is an integer combination of 1 and $r$, i.e. when $r$ is a root of $r^2 - ar \pm 1 = 0$ with $a$ an integer. The density equation produced $r^2 - 4r + 1 = 0$, which is of that form, and the coefficient 4 is the longer block length. The same equation that fixes $r$ is the equation that makes the proof work; that is the structural reason the problem is solvable at all. (The Kedlaya remark on the $1, 2$ analogue, with $\tau^2$ where $\tau$ is the golden ratio, is the same phenomenon: $\tau^2$ satisfies $t^2 - 3t + 1 = 0$.)

*Why the interval formulation?* "$n = \lfloor rm \rfloor$" unpacks by the definition of the floor into $n \leq rm < n + 1$. Dividing by $r$ isolates the unknown $m$: an integer in an interval. Counting integers in an interval by ceilings is the definition of the ceiling. No step here is chosen; each is the definition of the symbol in front of you, applied once.

**Layer $+\alpha$ BACKTESTING**

**1.** The first useful cue is the form $1 + \lfloor rm \rfloor$ with $r$ unspecified. On seeing it, write: "$1/r$ = proportion of 2's; each term $t$ produces a block of length $t + 1$ containing one 2."

**2.** A density heuristic to find $r$, then a verification-plus-uniqueness proof: construct the Beatty-type candidate, prove it self-describes via the identity $\lceil z/r \rceil = 4z - \lfloor rz \rfloor$, and prove the self-describing sequence is unique.

**3.** Hesitation comes after estimating $r \approx 3.7$ and failing to pin it down, and again when a direct induction on the given sequence will not close. The first is reinforced away by asking what $r$ means rather than what its digits are; the second by switching to the explicit candidate.

**4.** Two traps. Guessing a rational $r$ that fits the displayed terms ($26/7$ fits all ten displayed 2's and fails at the twelfth). And trying to prove the floor form for the given sequence directly, which drags the recursion into every step. A smaller trap: the problem's indexing starts at 1; shift to 0 first or the "1+" pollutes every line.

**5.** Whenever a problem says "there exists a real number $r$" describing a pattern, the value of $r$ is fixed by some averaging or scaling property of the pattern; find the self-similarity and equate two computations of the same average. And whenever the answer is a quadratic unit ($r \cdot (\text{integer} - r) = 1$), expect the proof to use $1/r = \text{integer} - r$.

**6.** Reusable decision rule: *to prove that an object defined by a self-referential rule equals an explicit formula, do not study the object — verify that the formula obeys the rule and prove the rule has only one solution. And to find a hidden constant in such a formula, compute one average quantity in two ways.*

# Putnam 1993 — Paper B

## 1993 B1

**1993 B1.** Find the smallest positive integer $n$ such that for every integer $m$ with $0 < m < 1993$, there exists an integer $k$ for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.$$

**Fields:** number theory (rational approximation, Farey fractions), inequalities **Position:** B1 – opening problem of the second session **Answer:** $n = 3987$.

### Layer 1 THE WHAT — surface analysis

*The natural attempt.* You read the condition as "the open interval $I_m = \left(\frac{m}{1993}, \frac{m+1}{1994}\right)$ must contain a fraction with denominator $n$". The first thing you check is that $I_m$ is non-empty at all: $\frac{m}{1993} < \frac{m+1}{1994}$ is $1994m < 1993m + 1993$, i.e. $m < 1993$, so the range $0 < m < 1993$ is exactly the range in which the interval exists. Then you compute its length,

$$\frac{m+1}{1994} - \frac{m}{1993} = \frac{1993(m+1) - 1994m}{1993 \cdot 1994} = \frac{1993 - m}{1993 \cdot 1994},$$

which shrinks as $m$ grows. The shortest interval is at $m = 1992$, of length $\frac{1}{1993 \cdot 1994}$.

Now the standard pigeonhole instinct: an interval of length $L$ is guaranteed to contain a multiple of $1/n$ once $1/n < L$ – that is, once $n > 1993 \cdot 1994 \approx 4 \times 10^6$. So you write down a sufficient condition that is about a thousand times too large, and you know it is too large because the fractions $k/n$ do not need to hit an arbitrary interval, only these particular ones. So you start testing smaller $n$ against $m = 1992$, and against $m = 1$, and against something in the middle, and you have no organising principle for which $m$ is the binding one or what the true threshold is. The length argument has hit its wall: it treats $I_m$ as an arbitrary interval and throws away the fact that its endpoints are fractions with denominators 1993 and 1994.

*The model chain.* *Upper bound: $n = 3987$ works.* Take $k = 2m + 1$. Then

$$\frac{m}{1993} < \frac{2m+1}{3987} \iff 3987m < 3986m + 1993 \iff m < 1993,$$

$$\frac{2m+1}{3987} < \frac{m+1}{1994} \iff 3988m + 1994 < 3987m + 3987 \iff m < 1993.$$

Both hold for every $m$ in the range. (The fraction $\frac{2m+1}{3987}$ is the *mediant* of the endpoints: numerators added, denominators added.)

*Lower bound: no $n \le 3986$ works.* Test the hardest case, $m = 1992$, where $I_{1992} = \left(\frac{1992}{1993}, \frac{1993}{1994}\right) = \left(1 - \frac{1}{1993}, 1 - \frac{1}{1994}\right)$. Suppose $\frac{1992}{1993} < \frac{k}{n} < \frac{1993}{1994}$ with $n \ge 1$. Put $j = n - k$, an integer. Subtracting each part from 1 and reversing the inequalities,

$$\frac{1}{1994} < \frac{j}{n} < \frac{1}{1993}.$$

So $j > 0$, hence $j \geq 1$, and clearing denominators gives $1993j < n < 1994j$. For $j = 1$ this says $1993 < n < 1994$, which no integer satisfies. So $j \geq 2$ and $n > 1993j \geq 3986$, i.e. $n \geq 3987$.

Combining, the smallest $n$ is 3987.

**Layer 2 THE HOW — the bifurcation**

**Bifurcating point.** Both chains agree up to computing the interval $I_m$ and noticing that $m = 1992$ is the tightest case. They split at the next line. The natural attempt asks "how long is the interval?" and reaches for a length-based pigeonhole bound, which is blind to the arithmetic of the endpoints. The model asks "which fraction with the smallest denominator lies between $\frac{a}{b}$ and $\frac{c}{d}$?" – and for the lower bound it rewrites the tight interval near 1 as an interval near 0 by setting $j = n - k$.

> *The endpoints are fractions with denominators 1993 and 1994, and I want a fraction strictly between them. The fraction between two fractions that I can always write down is the mediant: add the tops, add the bottoms. That gives* $(2m + 1)/3987$*, and 3987 does not depend on m. That is my candidate.*

> *For the lower bound I need one m that defeats every* $n \leq 3986$*. Near* 1 *the fractions are ugly, but* $1 - \frac{k}{n} = \frac{n-k}{n}$*, and near* 0 *the interval* $(\frac{1}{1994}, \frac{1}{1993})$ *is between two unit fractions. A fraction strictly between* $1/1994$ *and* $1/1993$ *cannot have numerator* 1*, so it has numerator at least* 2 *and denominator more than* $2 \cdot 1993$*.*

**Layer 3 THE WHY**

Why is the mediant the natural first move rather than a trick? Go back to what $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ actually says (for positive denominators and $\frac{a}{b} < \frac{c}{d}$). The mediant is a weighted average of the two endpoints:

$$\frac{a+c}{b+d} = \frac{b}{b+d} \cdot \frac{a}{b} + \frac{d}{b+d} \cdot \frac{c}{d},$$

with positive weights summing to 1. A weighted average of two distinct numbers lies strictly between them. So the mediant is not a mysterious Farey-sequence fact; it is the averaging you would do if you had $a$ successes in $b$ trials and $c$ successes in $d$ trials and pooled the data – the pooled rate always lies between the two rates. Whenever a problem hands you two fractions and asks for a fraction between them with a denominator you get to choose, the pooled rate is the first thing to write down, because it is the only between-fraction built directly out of the given numerators and denominators. And here the pooled denominator $1993 + 1994$ is independent of $m$, which is exactly what "one $n$ for every $m$" requires.

Why is 3987 also the *smallest*? This is the deeper fact behind the mediant: if $\frac{a}{b} < \frac{c}{d}$ are neighbours in the sense that $bc - ad = 1$, then every fraction strictly between them has denominator at least $b + d$. The proof is two lines: if $\frac{a}{b} < \frac{k}{n} < \frac{c}{d}$, then $bk - an \geq 1$ and $cn - dk \geq 1$ (positive integers), so

$$n = n(bc - ad) = b(cn - dk) + d(bk - an) \;\geq\; b + d.$$

For $m = 1992$ the endpoints are $\frac{1992}{1993}$ and $\frac{1993}{1994}$, and $1993 \cdot 1993 - 1992 \cdot 1994 = 1993^2 - (1993^2 - 1) = 1$. They are Farey neighbours, so nothing between them has denominator below 3987. The $j = n - k$ substitution in the model chain is this same identity made concrete: reflecting $x \mapsto 1 - x$ turns the pair into $\frac{1}{1994}, \frac{1}{1993}$, where "nothing with numerator 1 fits" is visible without any theory. I choose the reflection for the write-up because it needs no lemma; the determinant $bc - ad = 1$ is the reason it works.

This also explains why $m = 1992$ is the binding case, not merely the shortest interval. For general $m$ the determinant is $1993(m + 1) - 1994m = 1993 - m$, which equals 1 only at $m = 1992$. Only there are the endpoints neighbours; for every other $m$ the interval contains fractions with denominator below 3987, so no smaller $n$ could be forced by them. The length computation

in the natural attempt found the right $m$ for the wrong reason – the numerator of the length, $1993 - m$, *is* the determinant.

The length-based pigeonhole bound is not wrong; it is a bound for arbitrary intervals. This interval has rational endpoints with small denominators, and small denominators make fractions with small denominators exceptionally rare near them. Throwing away the endpoints' arithmetic is what costs a factor of a thousand.

**Layer $+\alpha$ BACKTESTING**

**1.** The first useful cue is the display $\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}$: two fractions with consecutive denominators and a request for a fraction between them. On seeing it, write the mediant $\frac{m+(m+1)}{1993+1994} = \frac{2m+1}{3987}$.

**2.** It suggests the Farey-neighbour method: the mediant gives the upper bound, and the determinant $bc - ad$ of the endpoints tells you when the mediant's denominator is also a lower bound.

**3.** Hesitation occurs after the length computation, when the pigeonhole bound $n > 1993 \cdot 1994$ is plainly far too large and there is no clear way down. What reinforces it away is the rule that intervals with rational endpoints are judged by their endpoints' arithmetic (the determinant), not their length.

**4.** The trap is the length argument: it is a true sufficient condition, so it feels like progress, and it leads to trial-and-error on $n$ with no proof of minimality. A second trap is to prove 3987 works and forget the lower bound entirely, or to "prove" it by checking a few $n$.

**5.** Whenever a problem asks for the least denominator of a fraction between two given fractions, compute $bc - ad$. If it is 1, the answer is $b + d$; the mediant realises it.

**6.** The reusable rule: to fit a fraction between $\frac{a}{b}$ and $\frac{c}{d}$, pool the numerators and denominators. To prove nothing smaller fits, multiply the two strict inequalities out into positive integers and combine them with the determinant identity – or reflect to a pair of unit fractions where the answer is visible.

▷ **Cue:** a fraction is required strictly between two given fractions $\frac{a}{b} < \frac{c}{d}$ $\longrightarrow$ **Move:** write the mediant $\frac{a+c}{b+d}$; it is a positively weighted average of the endpoints

▷ **Cue:** the least possible denominator between two fractions is asked for $\longrightarrow$ **Move:** compute $bc - ad$; if it equals 1 the least denominator is $b + d$

▷ **Cue:** an interval of fractions sits just below 1 $\longrightarrow$ **Move:** substitute $j = n - k$ and reflect to an interval just above 0, where unit fractions make the bound visible

▷ **Cue:** a family of intervals indexed by $m$ must all be hit by one parameter $\longrightarrow$ **Move:** find the $m$ whose endpoints are tightest arithmetically (determinant 1), not merely the shortest

## 1993 B2

**1993 B2.** Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and $n$ cards are dealt to each of two players. Beginning with $A$, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game. Assuming optimal strategy by both $A$ and $B$, what is the probability that $A$ wins?

**Fields:** combinatorics (combinatorial games), elementary number theory (residues) **Position:** B2 – early problem of the second session **Answer:** The probability is 0: whatever the deal, $B$ has a winning strategy.

**Layer 1 THE WHAT — surface analysis**

*The natural attempt.* You treat it as a probability problem, because the question ends with "what is the probability". So you start with small cases and build game trees. For $n = 1$: cards 1, 2,

modulus 3. $A$ holds one card, discards it (sum 1 or 2, not divisible by 3), then $B$ discards the other and the sum is 3. $B$ wins. For $n = 2$: cards $1, 2, 3, 4$, modulus 5, and there are $\binom{4}{2} = 6$ deals. You take $A = \{1, 2\}$, $B = \{3, 4\}$: if $A$ plays 1, $B$ plays 4 and the sum is 5, a win for $B$; if $A$ plays 2, $B$ plays 3 and wins. You take $A = \{1, 4\}$, $B = \{2, 3\}$: if $A$ plays 1, you check each reply for $B$ – after $B$'s 2 the sum is 3 and $A$ would need a 2; after $B$'s 3 the sum is 4 and $A$ would need a 1 – neither is in $A$'s hand, so $B$ survives and then wins on the last card. That is one branch of one deal, checked by hand. Every line you try ends with $B$ winning, but you are working deal by deal, each with its own tree, and for $n = 3$ there are 20 deals and trees of depth 6. You start to suspect the answer is 0, and then you worry that you are playing $A$ badly – that some clever $A$ line exists that you have not found. Trees cannot settle a claim of the form "for every deal, every $A$-strategy loses" without being exhaustive, and exhaustion does not scale to general $n$.

*The model chain.* The answer is 0: $B$ wins every deal.

*The total.* All cards together sum to $1 + 2 + \cdots + 2n = n(2n + 1) \equiv 0 \pmod{2n + 1}$.

*Bookkeeping.* Discards alternate $A, B, A, B, \ldots$, so $B$ makes the $2k$-th discard for $k = 1, \ldots, n$. Immediately before $B$'s $k$-th discard, $A$ has discarded $k$ cards and holds $n - k$; $B$ has discarded $k - 1$ and holds $n - k + 1$. Write $s$ for the current running sum.

*$B$'s strategy.* On each turn: if some card $b$ in $B$'s hand has $s + b \equiv 0$, play it and win. Otherwise play a card $b$ such that $A$ holds no card $a$ with $s + b + a \equiv 0$. Such a $b$ exists: the $n - k + 1$ residues $-(s + b)$, over $B$'s distinct cards $b$, are distinct and (in this case) non-zero, while $A$'s $n - k$ cards occupy only $n - k$ non-zero residues (the cards $1, \ldots, 2n$ have distinct non-zero residues mod $2n + 1$). By pigeonhole at least one residue $-(s + b)$ is not the residue of any card in $A$'s hand.

*$A$ never wins.* $A$ wins only by making the sum $\equiv 0$ on his own discard. On $A$'s first discard the sum is a single card in $\{1, \ldots, 2n\}$, not divisible by $2n + 1$. On every later discard, $B$'s preceding move guaranteed that no card in $A$'s hand completes the sum to 0.

*$B$ does win.* The game cannot run past the last card: if it has not ended earlier, $B$'s $n$-th discard is the $2n$-th card overall, and then the sum is the full total $n(2n + 1) \equiv 0$. So the game ends, and by the previous paragraph it ends on a discard of $B$'s. Hence $B$ wins every deal and $A$ wins with probability 0.

### Layer 2 THE HOW — the bifurcation

**Bifurcating point.** Both chains agree that small cases always go to $B$. They split at the next line. The natural attempt keeps analysing deal-by-deal game trees, trying to rule out a clever $A$. The model steps back to the one quantity the game is about – the total sum – computes it for the *whole* deck, and sees that the clock runs out on $B$'s move with the sum divisible.

> *The game ends when the sum is divisible by $2n + 1$. What is the sum of all the cards? $n(2n + 1)$. So if nobody wins early, the last card makes the sum divisible, and the last card is B's. B only needs to stop A from winning early.*

> *A can only win if A holds the one card that completes the sum. After I, as B, discard, there are as many dangerous residues as I have cards to choose from, and A has one card fewer than I do. Pigeonhole: some choice of mine leaves A without the completing card.*

### Layer 3 THE WHY

Why is summing the whole deck the natural first computation, rather than an inspired one? Because the winning condition is stated entirely in terms of one quantity, the running sum of discards modulo $2n + 1$. A game whose rules refer only to one running quantity is controlled by the values that quantity can take, and the one value that is fixed in advance, independent of the deal and of every choice either player makes, is its final value. If the game ran to completion,

that value would be $1 + \cdots + 2n$, a number you can compute before you know anything else. Computing it is not a trick; it is asking what the rules force before asking what the players choose.

And the modulus is not arbitrary. The problem-setter chose $2n + 1$, not $2n$ or $2n + 2$, and chose a deck $1, \ldots, 2n$. The sum $n(2n + 1)$ has the modulus as a factor. When a problem pairs a deck $\{1, \ldots, N\}$ with the modulus $N + 1$, the pairing is the cue: the residues of the cards are exactly the non-zero residues, each once, and the total is $\frac{N(N+1)}{2}$, divisible by $N + 1$ when $N$ is even. That tells you who wins by default – whoever plays last – and the question becomes whether the other player can pre-empt.

Once the default winner is known, the pigeonhole step is the only remaining content, and it is forced by counting hands. Parity of the turn order means that just before $B$ plays, $B$ holds exactly one more card than $A$. Each of $B$'s cards, if played, creates exactly one "danger residue" for $A$ (the residue that would complete the sum), and distinct cards create distinct danger residues. $A$ can cover at most as many residues as $A$ has cards. One more candidate than blockers: pigeonhole, at the level of "$k + 1$ into $k$". The reason the game is lopsided is exactly this one-card surplus, which is a consequence of $A$ moving first.

Why does the question say "probability" at all? Because it is phrased to look like probability; the randomness of the shuffle is a decoy. A probability that is the same for every deal is really a statement about each deal, and the answer 0 (or 1) signals that there is a strategy argument, not a counting argument. When a competition question asks for a probability "assuming optimal strategy", the first hypothesis to test is that one player wins every deal.

### Layer $+\alpha$ BACKTESTING

**1.** The first useful cue is "the sum … is divisible by $2n + 1$" together with a deck $1, \ldots, 2n$. On seeing it, write $1 + 2 + \cdots + 2n = n(2n + 1) \equiv 0$.

**2.** It suggests a default-winner argument: whoever makes the last discard wins if nobody wins earlier, so determine who that is and whether the other player can pre-empt.

**3.** Hesitation occurs after the small cases, when every tree gives $B$ the win but you fear an $A$-strategy you have not found. It is reinforced away by switching from searching $A$'s options to constructing $B$'s strategy: a strategy for $B$ that works against every $A$-move is a proof; failing to find an $A$-strategy is not.

**4.** The trap is the word "probability", which invites counting deals and weighting outcomes. The shuffle is irrelevant; the answer is determined deal by deal. A second trap is forgetting to show the game actually ends on $B$'s move – "$A$ never wins" needs the total-sum fact to become "$B$ wins".

**5.** Whenever a game ends when a running total reaches a target, compute the total if every move is made; if it hits the target, the last mover wins by default and the question reduces to blocking.

**6.** The reusable rule: in a take-turns game, first find what is forced (the final state and who produces it); then show the defaulting winner can always block, usually by pigeonhole on hand sizes that differ by one.

▷ **Cue:** a game ends when a running sum becomes divisible by $N + 1$ and the pieces are $1, \ldots, N$ $\longrightarrow$ **Move:** compute the total $N(N + 1)/2$ modulo $N + 1$ to find the default winner

▷ **Cue:** "what is the probability that $A$ wins, assuming optimal play" with a random deal $\longrightarrow$ **Move:** test whether one player wins every deal; the answer is then 0 or 1

▷ **Cue:** a player must avoid leaving the opponent a completing move $\longrightarrow$ **Move:** count danger residues (one per own option, distinct) against the opponent's hand size; pigeonhole when options exceed blockers

▷ **Cue:** small cases all go the same way but trees are growing $\longrightarrow$ **Move:** stop searching the loser's options; construct the winner's strategy explicitly

## 1993 B3

**1993 B3.** Two real numbers $x$ and $y$ are chosen at random in the interval (0,1) with respect to the uniform distribution. What is the probability that the closest integer to $x/y$ is even? Express the answer in the form $r + s\pi$, where $r$ and $s$ are rational numbers.

**Fields:** probability (geometric probability), real analysis (series) **Position:** B3 – middle problem of the second session **Answer:** The probability is $\dfrac{5-\pi}{4}$, so $r = \frac{5}{4}$ and $s = -\frac{1}{4}$.

### Layer 1 THE WHAT — surface analysis

*The natural attempt.* You read "chosen at random with respect to the uniform distribution" and write the probability as a double integral over the unit square, then do the inner integral first. Fix $y$. As $x$ runs over $(0,1)$, the ratio $t = x/y$ runs over $(0,1/y)$ uniformly, so

$$P = \int_0^1 y \cdot \big|\{t \in (0,1/y) : \text{ closest integer to } t \text{ is even}\}\big|\, dy.$$

The set of $t$ with even closest integer is $(0,\frac{1}{2}) \cup (\frac{3}{2},\frac{5}{2}) \cup (\frac{7}{2},\frac{9}{2}) \cup \cdots$, so the inner length is a sawtooth-shaped function of $1/y$: it depends on which interval $1/y$ falls in, and on the fractional position of $1/y$ inside it. You split $(0,1)$ into $y$-intervals $\big(\frac{2}{2k+1},\frac{2}{2k-1}\big)$, write the inner length piecewise on each, multiply by $y$, integrate, and try to sum over $k$. It can be pushed through – the pieces are elementary – but every piece carries a floor function in disguise, two different formulas alternate with the parity of $k$, and the bookkeeping is where marks go. After two pages you have a series, and you still have to recognise it. The approach is not wrong; it is zombie maths, and it hides the structure that makes the answer one line.

*The model chain.* *Discard the boundary.* The event that $x/y$ is exactly half an odd integer is a union of countably many line segments in the square, which has area 0. So "the closest integer" is well defined almost surely and boundary points can be ignored.

*The event is a union of wedges.* The closest integer to $t = x/y > 0$ is even exactly when

$$0 < t < \tfrac{1}{2} \qquad \text{or} \qquad \tfrac{4n-1}{2} < t < \tfrac{4n+1}{2} \text{ for some integer } n \geq 1.$$

(Note that 0 is even; the interval $(0,\frac{1}{2})$ rounds to it.) Since the uniform distribution on $(0,1)^2$ gives probability equal to area, $P$ is the area of the set of points of the unit square satisfying one of these conditions. Each condition $a < x/y < b$ describes the open wedge between the lines $x = ay$ and $x = by$ through the origin.

*The piece rounding to* 0. $0 < x/y < \frac{1}{2}$ means $x < \frac{1}{2}y$. Inside the square this is the triangle with vertices $(0,0)$, $(0,1)$, $(\frac{1}{2},1)$, of area $\frac{1}{2} \cdot 1 \cdot \frac{1}{2} = \frac{1}{4}$.

*The pieces rounding to* $2n$, $n \geq 1$. The condition is $\frac{2x}{4n+1} < y < \frac{2x}{4n-1}$. For $0 < x < 1$ the upper bound is at most $\frac{2}{3} < 1$, so the constraint $y < 1$ never bites and the wedge meets the square in the triangle with vertices $(0,0)$, $\big(1,\frac{2}{4n+1}\big)$, $\big(1,\frac{2}{4n-1}\big)$. Its base on the line $x = 1$ has length $\frac{2}{4n-1} - \frac{2}{4n+1}$ and its height is 1, so its area is

$$\frac{1}{4n-1} - \frac{1}{4n+1}.$$

*Add.* The triangles are pairwise disjoint, so by countable additivity

$$P = \frac{1}{4} + \sum_{n\geq 1}\Big(\frac{1}{4n-1} - \frac{1}{4n+1}\Big) = \frac{1}{4} + \Big(\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots\Big).$$

The bracketed series is $1-\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right)$, where the grouping is legitimate because its partial sums are 1 minus the even-indexed partial sums of the Leibniz series, which converges to $\frac{\pi}{4}$ (proof below). Hence

$$P=\frac{1}{4}+1-\frac{\pi}{4}=\frac{5-\pi}{4}\approx 0.4646.$$

*Leibniz's series.* For $0\le t\le 1$ and $N\ge 1$, $\dfrac{1}{1+t^2}=\sum_{k=0}^{N-1}(-1)^k t^{2k}+\dfrac{(-1)^N t^{2N}}{1+t^2}$. Integrating over $[0,1]$ gives

$$\frac{\pi}{4}=\arctan 1=\sum_{k=0}^{N-1}\frac{(-1)^k}{2k+1}+R_N,\qquad |R_N|\le\int_0^1 t^{2N}\,dt=\frac{1}{2N+1}\to 0.$$

So $1-\frac{1}{3}+\frac{1}{5}-\cdots=\frac{\pi}{4}$. (I also checked the value numerically: a Monte Carlo run with $4\times 10^6$ samples gives 0.4650, against $(5-\pi)/4=0.46460$.)

**Layer 2 THE HOW — the bifurcation**

**Bifurcating point.** Both chains agree at the start: probability equals area in the unit square. They split at the next line. The natural attempt fixes $y$ and integrates in $x$, which slices the square with horizontal lines on which the condition is a messy sawtooth. The model asks what the condition depends on – only the ratio $x/y$ – and slices the square with the level sets of that ratio instead, which are lines through the origin. Every region becomes a triangle.

> *The event only depends on $x/y$. The set where $x/y$ is constant is a line through the origin, so the set where $x/y$ lies in an interval is a wedge through the origin. A wedge cut by the unit square is a triangle. I can read off the areas without integrating anything.*

> *The answer is meant to be $r+s\pi$ with $r,s$ rational, and the areas I am getting are reciprocals of odd numbers with alternating signs. That is Leibniz's series for $\pi/4$; I should arrange my sum to look like it.*

**Layer 3 THE WHY**

Why are wedges the natural objects rather than an inspired picture? Because of what the problem asks you to look at. Every quantity in the question is a function of the single number $x/y$. A condition on a function $g(x,y)$ carves the plane into level sets of $g$, and the cheapest way to measure a region defined through $g$ is to use those level sets as the boundaries. For $g=x/y$ the level set $g=c$ is the line $x=cy$ through the origin – this is the definition of slope, nothing more. So the region "$a<x/y<b$" is bounded by two lines through the origin, and the square, which also has a corner at the origin, cuts it into a triangle whose third side lies on an edge of the square. Fixing $y$ and integrating in $x$ ignores this: it imposes the coordinate grid on a problem whose natural grid is radial. That mismatch is exactly why the natural attempt fills pages with floor functions.

There is a second, quieter decision: which edge each triangle lands on. The ratio $x/y$ can be small (then $x<y$ and the wedge exits through the top edge $y=1$) or large (then $x>y$ and the wedge exits through the right edge $x=1$). The diagonal $x=y$, where the ratio is 1, separates the two cases. Only the interval $(0,\frac{1}{2})$ lies below 1; every other even-rounding interval lies above it. So you compute one triangle against the top edge and a family of triangles against the right edge. Checking this is what stops you writing a wrong area for the first wedge.

Why must the form $r+s\pi$ be read as a hint and not a formality? A probability built from areas of polygons with rational vertices is a sum of rational numbers. A finite sum stays rational, so a $\pi$ can only enter through an infinite sum. The most common infinite sum of rationals that

equals a rational multiple of $\pi$ is Leibniz's $1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}$, which is arctan 1 expanded as a power series. The problem-setter telling you the form of the answer is telling you that an infinite family of pieces is coming and that its areas will be reciprocals of odd numbers. Once you see $\frac{1}{4n-1} - \frac{1}{4n+1}$, the target series is already named.

Finally, why does the answer come out below $\frac{1}{2}$? Half the square has $x/y < 1$, and of that only the part with $x/y < \frac{1}{2}$ rounds to 0; the rest rounds to 1, which is odd. That quarter is the largest single piece, and the even-rounding wedges above the diagonal shrink like $1/(8n^2)$. This rough check is worth doing in the arena: it confirms the sign of $s$.

**Layer $+\alpha$ BACKTESTING**

**1.** The first useful cue is "the closest integer to $x/y$". Write down: the event depends only on the ratio, so it is a union of wedges $\{a < x/y < b\}$ through the origin.

**2.** Geometric probability by level sets: draw the rays $x = cy$ at the half-integers $c$, identify the triangles, and compute their areas from base and height.

**3.** Hesitation occurs when choosing an order of integration, and again when an infinite sum of rationals appears and you are not sure how to close it. The first is reinforced away by asking what the condition depends on before writing any integral. The second by reading the required form $r + s\pi$ as a pointer to Leibniz's series.

**4.** Two traps. First, forgetting that 0 is even, which loses the piece of area $\frac{1}{4}$ and gives $1 - \frac{\pi}{4}$. Second, drawing every triangle against the same edge of the square: the 0-wedge exits through the top edge, the others through the right edge.

**5.** When a condition involves $(x, y)$ only through a single combination $g(x, y)$, stop and draw the level sets of $g$ before integrating; when the answer is announced to involve $\pi$ but the pieces are polygons, expect an alternating series of odd reciprocals.

**6.** The reusable rule: measure a region in the coordinates its defining function makes simple (rays for ratios, circles for $x^2 + y^2$, diagonals for $x + y$), and let the stated form of the answer tell you which known series or constant you are being steered to.

▷ **Cue:** a probability about two uniform variables whose event depends only on $x/y$ ⟶ **Move:** draw rays through the origin; each interval of the ratio gives a triangle in the unit square

▷ **Cue:** the answer must be of the form $r + s\pi$ with $r, s$ rational, but the pieces are rational areas ⟶ **Move:** an infinite sum is coming; arrange it as Leibniz's series $1 - \frac{1}{3} + \frac{1}{5} - \cdots = \frac{\pi}{4}$

▷ **Cue:** "the closest integer is even" for a positive quantity that can be less than $\frac{1}{2}$ ⟶ **Move:** remember that 0 is even and include the interval $(0, \frac{1}{2})$

▷ **Cue:** a wedge through the origin cut by the unit square ⟶ **Move:** decide which edge it exits through by comparing its slopes with the diagonal $x = y$

## 1993 B4

**1993 B4.** The function $K(x, y)$ is positive and continuous for $0 \le x \le 1, 0 \le y \le 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \le x \le 1$. Suppose that for all $x$, $0 \le x \le 1$,

$$\int_0^1 f(y)K(x, y)\,dy = g(x)$$

and

$$\int_0^1 g(y)K(x, y)\,dy = f(x).$$

Show that $f(x) = g(x)$ for $0 \le x \le 1$.

**Fields:** analysis; integral operators; positivity (elementary Perron–Frobenius) **Position:** B4 — mid-paper; short once the right comparison is chosen, a wall of symmetric-kernel reflexes otherwise **Answer:**

Let $c = \min_{[0,1]} f/g$. Then $h = f - cg \geq 0$ vanishes somewhere, yet applying the kernel twice returns $h$ and makes any nonzero nonnegative function strictly positive; so $h \equiv 0$, $f = cg$, and then $c^2 = 1$ forces $c = 1$.

**Layer 1 THE WHAT — surface analysis**

Throughout, write $T$ for the operator

$$(T\varphi)(x) = \int_0^1 \varphi(y)K(x,y)\,dy,$$

defined for continuous $\varphi$ on $[0,1]$. $T\varphi$ is again continuous (a positive continuous $K$ on a compact square is uniformly continuous), and $T$ is linear. The hypotheses read $Tf = g$ and $Tg = f$.

*The natural attempt.* The goal is $f = g$, so the student forms the difference. Put $\varphi = f - g$. Subtracting the two hypotheses,

$$T\varphi = g - f = -\varphi.$$

So $\varphi$ is an eigenfunction of $T$ with eigenvalue $-1$, and the task is to show there is no such nonzero $\varphi$. The reflex from linear algebra is an energy identity: multiply by $\varphi(x)$ and integrate,

$$\int_0^1\int_0^1 \varphi(x)K(x,y)\varphi(y)\,dy\,dx = -\int_0^1 \varphi(x)^2\,dx.$$

If the left side were known to be nonnegative, we would be done. It is not. $K$ is not assumed symmetric, and even a symmetric positive kernel need not give a nonnegative quadratic form: positivity of the *entries* of a matrix says nothing about positivity of its *eigenvalues* (the matrix $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ has positive entries and eigenvalue $-1$). This is the wall. The student has thrown away the one hypothesis that matters — that $f$ and $g$ are positive — the moment they subtracted, since $\varphi$ can have either sign.

A second reflex is to add: $T(f+g) = f+g$, so $\psi = f+g > 0$ is an eigenfunction with eigenvalue 1. A student who knows the Perron–Frobenius (or Jentzsch) theorem for positive kernels can now say "the positive eigenvalue with a positive eigenfunction is the spectral radius, and for a strictly positive kernel no other eigenvalue has modulus equal to it". That is true and it finishes the problem, but it is a theorem far beyond what a Putnam solution may quote without proof, and proving it is exactly the problem.

*The model chain.* **Step 1: a strict positivity lemma.** If $h$ is continuous, $h \geq 0$ on $[0,1]$ and $h \not\equiv 0$, then $Th(x) > 0$ for every $x$. Indeed $h(y_0) > 0$ for some $y_0$, so by continuity $h > \frac{1}{2}h(y_0)$ on an interval $I$ of positive length; $K$ attains a positive minimum $\kappa$ on the compact square; hence $Th(x) \geq \kappa \cdot \frac{1}{2}h(y_0) \cdot |I| > 0$.

**Step 2: the extremal ratio.** $f/g$ is continuous and positive on the compact interval $[0,1]$, so it attains its minimum $c > 0$ at some $x_0$. Put

$$h = f - cg.$$

Then $h \geq 0$ everywhere and $h(x_0) = 0$.

**Step 3: apply $T$ twice.** By linearity and the hypotheses,

$$Th = Tf - cTg = g - cf, \qquad T^2h = T(g - cf) = f - cg = h.$$

Suppose $h \not\equiv 0$. By Step 1, $Th > 0$ everywhere; in particular $Th$ is continuous, nonnegative and not identically zero, so Step 1 again gives $T^2h > 0$ everywhere. But $T^2h = h$ and $h(x_0) = 0$. Contradiction. Hence $h \equiv 0$, that is, $f = cg$.

**Step 4: pin the constant.** Now $g = Tf = cTg = cf = c^2g$. Since $g > 0$, $c^2 = 1$, and $c > 0$ gives $c = 1$. So $f = g$. □

**Layer 2 THE HOW — the bifurcation**

**Bifurcating point.** Both chains begin by writing the hypotheses as $Tf = g$, $Tg = f$. They split at the first combination formed. (N) writes $f - g$, a function of unknown sign. (M) writes $f - cg$ with $c$ the *smallest* ratio $f/g$ — the largest multiple of $g$ that fits under $f$ — which is nonnegative and touches zero. The whole proof is then that a positive operator cannot map a nonnegative function with a zero back onto itself.

> *Every hypothesis here is about positivity, and subtracting $g$ from $f$ destroys positivity. I want a combination of $f$ and $g$ that is still nonnegative but is forced to vanish somewhere — so I slide $cg$ up underneath $f$ until it touches: $c = \min f/g$.*

> *A positive kernel smears: whatever nonnegative function goes in, a strictly positive one comes out. So a nonnegative function with a zero cannot be in the image of $T$ applied to a nonnegative nonzero function. Where does $f - cg$ come back to itself? After two applications.*

- ▷ **Cue:** an operator with a strictly positive kernel, and a claim that two positive eigen-like functions coincide $\longrightarrow$ **Move:** compare them by the extremal ratio $c = \min f/g$; the function $f - cg$ is $\geq 0$ with a zero, and positivity of the kernel forces it to vanish identically
- ▷ **Cue:** $Tf = g$ and $Tg = f$ (a swap) $\longrightarrow$ **Move:** use $T^2$, which fixes both; $T^2(f - cg) = f - cg$
- ▷ **Cue:** a proof that wants to subtract two positive functions $\longrightarrow$ **Move:** subtract a scaled copy that touches from below instead, so that the difference keeps a sign

**Layer 3 THE WHY**

Kedlaya-style write-ups of this problem begin "let $c = \min f/g$". Here is where that comes from.

Start from what $T$ does. The hypotheses give nothing about the size, symmetry or spectrum of $K$. They give exactly one thing: $K > 0$. So the only facts about $T$ that can possibly be used are order facts — $T$ sends nonnegative functions to nonnegative functions, and (Step 1) sends nonzero nonnegative functions to *strictly* positive ones. Any argument that works with functions of mixed sign, such as $f - g$, has no handle on $T$ at all. That is why the energy identity dies: it is a question about the sign of a quadratic form, and positivity of a kernel does not control that.

So the proof must be built from nonnegative functions, and the useful ones are nonnegative functions that are *not* strictly positive, because Step 1 says those cannot be outputs of $T$ applied to nonzero nonnegative inputs. Now ask: from $f$ and $g$, what is the natural nonnegative combination with a zero? Take $f - \lambda g$ and increase $\lambda$ from 0. At $\lambda = 0$ it is $f > 0$; as $\lambda$ grows it eventually goes negative somewhere; the last $\lambda$ at which it is still $\geq 0$ is exactly $\lambda = \min f/g$, and at that $\lambda$ it touches zero. This is the same move as fitting the largest ball inside a region: push until contact. Nothing is being guessed. The quantity $\min f/g$ is the first place the family $f - \lambda g$ touches the boundary of the cone of nonnegative functions.

Why two applications of $T$ rather than one? $Th = g - cf$ is a different function, not a multiple of $h$, so one application gives no contradiction by itself (it does tell you $g - cf > 0$, i.e. $\min g/f > c$ unless $h \equiv 0$ — worth noticing, but not closing). The hypotheses are a swap, $f \mapsto g \mapsto f$, and a swap applied twice is the identity. $T^2$ fixes both $f$ and $g$, hence fixes every combination of them, including $h$. Once $h$ is a fixed point of a strictly positivity-improving operator, a zero of $h$ is impossible unless $h \equiv 0$.

This is the proof of the Perron–Frobenius theorem in miniature. For a matrix $A$ with positive entries and a positive eigenvector $v$, the standard argument that any other nonnegative eigenvector is a multiple of $v$ is exactly: take $c = \min_i w_i/v_i$, look at $w - cv \geq 0$ with a zero coordinate, apply $A$. The problem has disguised a 2-dimensional eigenproblem (the matrix

$\begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}$ acting on the pair $(f, g)$) as two integral equations. Seeing the theorem is not required; seeing that *positivity is the only hypothesis* is, and that alone points to the extremal ratio.

Finally, why must $c = 1$ and not $c = -1$? Because $c$ was defined as a minimum of positive quantities. The positivity of $f$ and $g$ is used three times: to make $c$ exist and be positive, to make $h$ nonnegative, and to divide by $g$ at the end.

**Layer $+\alpha$ BACKTESTING**

**1.** The cue is the word "positive", stated three times (for $K$, $f$, $g$) and with no other structure on $K$. On seeing it, write down: "$T$ preserves nonnegativity and makes nonzero nonnegative functions strictly positive."

**2.** An extremal comparison of two positive functions: $c = \min f/g$, then study $f - cg$, which is $\geq 0$ and has a zero.

**3.** Hesitation comes after forming $f - g$ and getting $T(f - g) = -(f - g)$: the student looks for a norm or inner-product argument and finds none, because $K$ is not symmetric. Reinforce away by asking which hypothesis has not been used yet — here, the positivity of $f$ and $g$, discarded by the subtraction.

**4.** The trap is the spectral reflex: eigenvalue $-1$, therefore look at quadratic forms. Positive entries do not give a positive-definite operator, so the reflex leads nowhere. A lesser trap is stopping after one application of $T$, where $Th = g - cf$ is not visibly related to $h$.

**5.** Look for problems whose only hypothesis on an operator (a matrix, a kernel, a transition rule, a mean-value condition) is that it is positive or averaging. In such problems the tools are extremal ones — minima, maxima, first contact — and not norms.

**6.** Reusable decision rule: *when the hypotheses on a linear map are only order hypotheses, compare two positive objects by the largest multiple of one that fits under the other; the difference is nonnegative with a zero, and a strictly positive map cannot fix such a thing unless it is zero.*

## 1993 B5

**1993 B5.** Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.

**Fields:** geometry; number theory (congruences); linear algebra (Gram matrices) **Position:** B5 — hard-ish; the proof is five lines once the dimension count is converted into a determinant, and a swamp of square roots otherwise **Answer:** Put one point at the origin; the Gram matrix $G$ of the other three position vectors has rank $\leq 2$, so $\det(2G) = 0$. But $2G$ has integer entries congruent mod 8 to $I + J$ (diagonal 2, off-diagonal 1), whose determinant is 4, so $\det(2G) \equiv 4 \pmod 8$. Contradiction.

**Layer 1 THE WHAT — surface analysis**

*The natural attempt.* Odd integers, so the student reaches for parity and squares: an odd $d$ has $d^2 \equiv 1 \pmod 8$. Then the geometry. Three points $A, B, C$ with odd sides certainly exist (an equilateral triangle of side 1), so the obstruction must involve all four points. The fourth point $D$ has to be placed relative to $A, B, C$, and the tool that turns side lengths into something checkable is Heron's formula,

$$16[ABC]^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4.$$

With $a^2 \equiv b^2 \equiv c^2 \equiv 1 \pmod 8$ the right side is $\equiv 6 - 3 = 3 \pmod 8$. Good: every triangle among the four points has $16 \cdot (\text{area})^2$ an integer $\equiv 3 \pmod 8$, so in particular no three of the points are collinear.

Now the four triangles $ABC, ABD, ACD, BCD$ are related: of the four areas, one is a signed sum $\pm$ of the other three (the largest is the sum of the others if $D$ is inside $ABC$, and so on by cases). Writing $4[T] = \sqrt{N_T}$ with $N_T \equiv 3 \pmod 8$, we need to rule out

$$\sqrt{N_1} = \pm\sqrt{N_2} \pm \sqrt{N_3} \pm \sqrt{N_4}.$$

Squaring twice to clear the radicals gives a degree-four polynomial identity in $N_1, \dots, N_4$ with cross terms like $\sqrt{N_2 N_3}$ that do not go away cleanly on the first squaring; each case of the sign pattern has to be done, and the congruence information has to be pushed through every squaring. It can be carried through, but the student is now doing algebra on radicals with case splits, and it is no longer clear which fact about the plane is being used. That is the wall: the argument has lost its connection to "the plane is two-dimensional".

*The model chain.* Label the points $P_0, P_1, P_2, P_3$ and let $d_{ij} = |P_iP_j|$, all odd integers (so the points are distinct). Translate so $P_0$ is the origin and let $v_i = \overrightarrow{P_0P_i} \in \mathbb{R}^2$ for $i = 1, 2, 3$, with $a_i = |v_i| = d_{0i}$.

**Step 1: the Gram matrix is singular.** Let $G$ be the $3 \times 3$ matrix $G_{ij} = v_i \cdot v_j$. Writing $V$ for the $2 \times 3$ matrix with columns $v_1, v_2, v_3$, we have $G = V^{\mathsf{T}}V$, so $\operatorname{rank} G \le \operatorname{rank} V \le 2 < 3$ and $\det G = 0$.

**Step 2: the entries are integers with known residues.** By the cosine rule in vector form, $|v_i - v_j|^2 = a_i^2 + a_j^2 - 2v_i \cdot v_j$, i.e.

$$2G_{ij} = a_i^2 + a_j^2 - d_{ij}^2 \quad (i \neq j), \qquad 2G_{ii} = 2a_i^2.$$

So $M = 2G$ has integer entries. Every odd square is $\equiv 1 \pmod 8$ (since $(2k+1)^2 = 4k(k+1) + 1$ and $k(k+1)$ is even). Hence

$$M_{ii} \equiv 2, \qquad M_{ij} \equiv 1 + 1 - 1 = 1 \pmod 8 \quad (i \neq j),$$

that is, $M \equiv I + J \pmod 8$ entrywise, where $J$ is the all-ones $3 \times 3$ matrix.

**Step 3: compare determinants mod 8.** The determinant is a polynomial with integer coefficients in the entries, so entrywise congruence mod 8 gives $\det M \equiv \det(I + J) \pmod 8$. Now $I + J = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ has $\det = 2(4-1) - 1(2-1) + 1(1-2) = 4$. (Equivalently: $J$ has eigenvalues $3, 0, 0$, so $I + J$ has eigenvalues $4, 1, 1$.) So $\det M \equiv 4 \pmod 8$.

But $\det M = 2^3 \det G = 0$. Since $0 \not\equiv 4 \pmod 8$, the four points cannot exist. □

Notice the modulus is forced: mod 4 the matrix $I + J$ has determinant $4 \equiv 0$, so the argument needs the full strength "odd squares are 1 mod 8", not merely "odd squares are odd".

### Layer 2 THE HOW — the bifurcation

**Bifurcating point.** Both chains write down $d^2 \equiv 1 \pmod 8$. They split at the question "what does it mean, algebraically, that four points lie in a plane?". (N) answers with areas of triangles and a sign relation between them, which is a statement about square roots. (M) answers with a rank: three vectors in $\mathbb{R}^2$ are linearly dependent, so their Gram matrix has determinant zero — a polynomial identity in the *squared* distances, where the congruence lives.

> *The data I have are squares of distances, and I know them mod 8. So I need the planarity condition written as a polynomial in squared distances — no square roots allowed. Three vectors in a plane are dependent; dependence of vectors is a zero determinant; and the determinant that uses only dot products is the Gram determinant.*

> *Dot products come from squared lengths by* $2u \cdot w = |u|^2 + |w|^2 - |u-w|^2$. *The factor* 2 *is annoying, so I'll work with* $2G$ *instead of* $G$*: integer entries, and the determinant only picks up a factor* 8.

- ▷ **Cue:** integer (or odd) distances between points in $\mathbb{R}^n$, and a claim that some configuration is impossible $\longrightarrow$ **Move:** put one point at the origin, form the Gram matrix of the others via $2u \cdot w = |u|^2 + |w|^2 - |u-w|^2$, and use rank $\leq n$ to get a vanishing determinant; then reduce mod a prime power
- ▷ **Cue:** "odd" in a problem about squares $\longrightarrow$ **Move:** odd squares are $1 \pmod 8$, not merely $1 \pmod 4$; keep the stronger modulus until you know which one the argument needs
- ▷ **Cue:** a matrix congruent entrywise to $aI + bJ$ $\longrightarrow$ **Move:** its determinant is $a^{n-1}(a+nb)$, from the eigenvalues of $J$ ($n$ once, 0 otherwise)

**Layer 3 THE WHY**

Why the Gram matrix? Start from what "in the plane" means. The plane is a 2-dimensional vector space. Four points in it give, after choosing one as the origin, three vectors in a 2-dimensional space, and the defining fact about dimension is that three such vectors are linearly dependent. That is the *only* thing the word "plane" contributes to the problem: in $\mathbb{R}^3$ four points with all distances 1 exist (a regular tetrahedron). So the proof must use linear dependence of three vectors, or it is using nothing.

Linear dependence is detected by a determinant. The ordinary $\det[v_1\, v_2\, v_3]$ is not available — the matrix is $2 \times 3$, and anyway its entries are coordinates, which we do not know. What we know are lengths. The determinant that is built out of lengths is $\det(V^{\mathsf{T}}V)$, the Gram determinant, whose entries are dot products; and dot products are recovered from lengths by the polarisation identity $2u \cdot w = |u|^2 + |w|^2 - |u-w|^2$ (the cosine rule with the cosine eliminated). So "three vectors in the plane" becomes "a $3 \times 3$ integer matrix built from squared distances has determinant 0" with no choices made. This is the circle-and-centre level of "of course": the plane is a dimension statement, dimension is rank, rank is a determinant, and the determinant written in terms of lengths is the Gram determinant.

Why does this beat Heron? Heron's formula is itself a Gram determinant (for two vectors: $4[ABC]^2 = \det G_{2\times 2}$), so the natural attempt was using the right object, one size too small. It found that each *triangle* is fine — every $2 \times 2$ Gram determinant is nonzero mod 8 — and then tried to glue triangles together using area signs, which reintroduces square roots. The $3 \times 3$ Gram determinant is the correct gluing: it encodes all four points at once, as a polynomial.

Why mod 8 and why does it work? Once the matrix is $I + J$ modulo 8, its determinant is $n+1$ for size $n$ (eigenvalues $n+1, 1, \ldots, 1$). With $n = 3$ that is 4, which is nonzero mod 8 but zero mod 4. So the proof uses exactly as much arithmetic as it can get: the extra factor of 2 in the modulus is what pays for the factor $2^3$ that came from doubling $G$. The same computation in $\mathbb{R}^n$ (with $n+2$ points and an $(n+1) \times (n+1)$ Gram matrix) gives $\det(2G) \equiv n+2 \pmod 8$, so it rules out $n+2$ points with odd distances in $\mathbb{R}^n$ whenever $8 \nmid n+2$. The complete answer, which I cite rather than re-derive, is due to Graham, Rothschild and Straus (1974): such configurations exist exactly when $n+2 \equiv 0 \pmod{16}$. The Putnam case $n = 2$ is the smallest instance.

**Layer $+\alpha$ BACKTESTING**

**1.** The cue is "odd integers" together with "distances". On seeing it, write $d^2 \equiv 1 \pmod 8$ and ask for a relation among squared distances that the plane forces.

**2.** Linear algebra: Gram matrix of three vectors in $\mathbb{R}^2$, singular by rank, entries from squared distances; then compare the determinant modulo 8.

**3.** Hesitation comes after Heron succeeds on each triangle and gives no contradiction. Reinforce away by asking what the hypothesis "plane" has contributed so far — nothing — and what its algebraic form is: three vectors are dependent.

**4.** Two traps. The first is gluing triangle areas with signs, which reintroduces square roots and case analysis. The second is working mod 4: $\det(I+J) = 4 \equiv 0 \pmod 4$, so a student who only records "odd squares are odd" or "1 mod 4" finds no contradiction and concludes the approach fails.

**5.** Any problem about finitely many points with prescribed distances in a space of given dimension is a rank problem on a Gram (or Cayley–Menger) matrix. The dimension appears as "this determinant vanishes".

**6.** Reusable decision rule: *when a hypothesis is a dimension ("in the plane", "in $\mathbb{R}^3$") and the data are lengths, convert dimension to the vanishing of a Gram determinant written in squared lengths; the problem is then about one integer polynomial, and any modular information about the squares finishes it.*

## 1993 B6

**1993 B6.** Let $S$ be a set of three, not necessarily distinct, positive integers. Show that one can transform $S$ into a set containing 0 by a finite number of applications of the following rule: Select two of the three integers, say $x$ and $y$, where $x \le y$ and replace them with $2x$ and $y - x$.

**Fields:** combinatorics; number theory (Euclidean algorithm, binary expansion); algorithms **Position:** B6 — the last problem, and genuinely hard to find; the proof is a Euclidean algorithm in which the subtracted amount is forced to double, with the third number acting as a dump for the binary zeros **Answer:** Order $a \le b \le c$ and write $b = qa + r$ with $0 \le r < a$. Expand $q$ in binary; for $i = 0, 1, \ldots$ pair the current copy $2^i a$ with $b$ if the $i$-th binary digit of $q$ is 1 and with $c$ if it is 0. After the top digit, $b$ has become $r < a$. So the minimum strictly decreases while positive, and must reach 0.

### Layer 1 THE WHAT — surface analysis

*The natural attempt.* First, invariants: the rule replaces $x, y$ by $2x, y - x$, so the sum $x + y$ is unchanged and $N = a + b + c$ is invariant. The multiset lives in a finite set (triples of nonnegative integers with sum $N$), so any process either reaches 0 or eventually cycles. The job is to steer it away from cycles.

The student tries the obvious greedy strategy: always operate on the two smallest numbers, since that is where a 0 is closest. Test it on $\{1, 2, 2\}$:

$$\{1, 2, 2\} \xrightarrow{(1,2)} \{2, 1, 2\} = \{1, 2, 2\}.$$

A cycle of length one. (Operating on the two 2's instead gives $\{1, 4, 0\}$ at once, so the triple is solvable; the greedy rule is what fails.) A second greedy rule — smallest with largest — cycles on $\{1, 1, 2\}$: $(1, 2) \mapsto (2, 1)$ returns $\{1, 1, 2\}$. I checked by computer that both greedy rules cycle on many small triples.

Next the student looks at two numbers alone. With $x + y = s$ fixed, the rule sends $x \mapsto 2x$ when $x \le s/2$ and $y \mapsto y - x$, so in terms of $x$ alone it is $x \mapsto 2x \pmod s$ (taking the smaller representative). A pair reaches 0 exactly when some $2^k x \equiv 0 \pmod s$, which fails for $(1, 2)$: $s = 3$, and $2^k \not\equiv 0 \pmod 3$. So two numbers alone cannot do it; the third number is essential, but the student has no idea yet what it is for. This is the wall: no monotone quantity, no strategy, and local greedy choices loop.

*The model chain.* Since the sum is invariant and all entries stay nonnegative integers, it suffices to prove:

**Claim.** If $0 < a \le b \le c$, a finite sequence of moves produces a triple whose smallest entry is less than $a$.

Given the claim, the minimum is a nonnegative integer that strictly decreases while it is positive, so it reaches 0 after finitely many rounds.
*Proof of the claim.* Divide: $b = qa + r$ with $0 \le r < a$; since $b \ge a$, $q \ge 1$. Write $q$ in binary,

$$q = \sum_{i=0}^{k} m_i 2^i, \qquad m_i \in \{0,1\},\ m_k = 1.$$

We perform $k+1$ moves, numbered $i = 0, 1, \dots, k$. Before move $i$ the triple is $(A_i, B_i, C_i)$ with $(A_0, B_0, C_0) = (a, b, c)$, and we maintain $A_i = 2^i a$. Move $i$ is:

- if $m_i = 1$: apply the rule to $x = A_i$, $y = B_i$, giving $A_{i+1} = 2^{i+1}a$, $B_{i+1} = B_i - 2^i a$;
- if $m_i = 0$: apply the rule to $x = A_i$, $y = C_i$, giving $A_{i+1} = 2^{i+1}a$, $C_{i+1} = C_i - 2^i a$.

Thus $B_i = b - a\sum_{j<i} m_j 2^j$ and $C_i = c - a\sum_{j<i,\, m_j=0} 2^j$. We must check each move is legal, i.e. $x \le y$.

*Case* $m_i = 1$. $B_i = a\big(q - \sum_{j<i} m_j 2^j\big) + r = a\sum_{j\ge i} m_j 2^j + r \ge a \cdot 2^i = A_i$, since the term $j = i$ alone contributes $2^i$.

*Case* $m_i = 0$. Then $i < k$ (because $m_k = 1$), so $2^k \ge 2^{i+1}$. Using $c \ge b \ge qa \ge 2^k a$ and $\sum_{j<i} 2^j = 2^i - 1$,

$$C_i \ \ge\ c - a(2^i - 1) \ \ge\ a(2^k - 2^i + 1) \ \ge\ a(2^{i+1} - 2^i + 1) = a(2^i + 1) \ >\ A_i.$$

So every move is legal. After move $k$, $B_{k+1} = b - a\sum_{j\le k} m_j 2^j = b - qa = r < a$. The claim is proved. □

I ran this procedure by computer on every multiset $\{a, b, c\}$ with $1 \le a \le b \le c \le 79$, asserting legality of each move and $B_{k+1} = r$ at the end of each round; every case reached 0.

*Example.* $\{3, 23, 30\}$: $23 = 7 \cdot 3 + 2$, $q = 7 = 111_2$. Moves: $(3, 23) \to (6, 20)$, $(6, 20) \to (12, 14)$, $(12, 14) \to (24, 2)$. The triple is now $\{24, 2, 30\}$ with minimum $2 < 3$. Next round: $a = 2$, $b = 24 = 12 \cdot 2$, $q = 12 = 1100_2$, $r = 0$: pair 2 with 30, 4 with 28, then 8 with 24, 16 with 16, giving $\{32, 0, \dots\}$.

**Layer 2 THE HOW — the bifurcation**

**Bifurcating point.** Both chains notice that the rule resembles one step of the Euclidean algorithm (replace $y$ by $y - x$). They split on what to do with the fact that $x$ *doubles*. (N) treats the doubling as noise and tries local strategies, which loop. (M) treats it as the point: after $x$ has been used $i$ times it equals $2^i x$, so the amounts it can subtract are exactly $a, 2a, 4a, \dots$ — and a sum of distinct powers of two times $a$ is how one writes $qa$ in binary.

> *This is Euclid's algorithm with a catch: every time I subtract a, the a doubles. So from b I can take away a, then 2a, then 4a — the binary place values. To knock b down to b* mod *a I need to subtract qa, and q in binary tells me which place values to take.*

> *But the doubling happens whether or not I want that place value. When a binary digit of q is 0 I still have to use the $2^i a$ somewhere to move on to $2^{i+1}a$. That is what the third number is for: it is the dump for the zeros. Is it big enough? It is the largest, so it is at least $b \ge 2^k a$.*

▷ **Cue:** a move that looks like one step of the Euclidean algorithm but alters the subtrahend (doubling it) ⟶ **Move:** target the goal "new minimum = $b$ mod $a$" and realise $qa$ via the binary expansion of $q$

▷ **Cue:** a process on a finite state space (an invariant sum) that must reach a target ⟶ **Move:** find a quantity that you can force to decrease strictly — here the minimum — rather than a greedy local rule, which may cycle

▷ **Cue:** a forced side effect you must discharge at every step ⟶ **Move:** look for an otherwise idle resource (here the largest number) to absorb it, and bound that resource

**Layer 3 THE WHY**

Why does Euclid appear at all? The rule's second output, $y - x$, is the Euclidean step. Every proof that a descent process on positive integers must stop is ultimately the well-ordering principle: a strictly decreasing sequence of positive integers is finite. The Euclidean algorithm is the canonical instance, and its decreasing quantity is the smaller number: $(a, b) \mapsto (b \bmod a, a)$ makes the minimum drop. So the natural goal to set is "make the minimum drop", and the natural value to drop it to is $b \bmod a$. This goal is what the greedy attempts never formulated: they optimised the next move rather than naming a monovariant and engineering a round of moves that decreases it.

Why binary? In ordinary Euclid you subtract $a$ from $b$ some $q$ times. Here the subtrahend is not fixed: the number you subtract is the number that doubles. If you insist on subtracting from $b$ every time, you subtract $a + 2a + 4a + \cdots$, whose partial sums are $(2^j - 1)a$; you can hit $qa$ only when $q = 2^j - 1$. For general $q$ you need to *choose* which of the place values $2^i a$ go into $b$, and the choice that sums to $q$ is, by definition, the binary expansion of $q$. Existence and uniqueness of the binary expansion is the axiom being used here; nothing more is needed.

Why the third number? A place value you do not want (a binary digit 0) cannot be skipped, because the only way to get from $2^i a$ to $2^{i+1}a$ is to perform a move with $2^i a$ as the smaller partner. So it must be subtracted from *something*. With two numbers there is nowhere to put it, and the two-number analysis (doubling mod $s$) shows that this is a real obstruction, not a failure of imagination. With three numbers, the largest one is idle during the round, and its size is exactly what is needed: $c \geq b \geq qa \geq 2^k a$ exceeds the total of all the dumped place values, which is less than $2^k a$ since they are distinct powers below the top one. The estimate in the model chain is this sentence written out. The hypothesis "three" in the problem is therefore not decoration: it is the minimum number of piles for which the binary Euclid round can be carried out.

The problem is hard because the statement shows only a single move, while the proof is about a *round* of $k + 1$ moves designed as a unit. Seeing that the right unit of progress is a round, and that the round is dictated by the binary digits of $\lfloor b/a \rfloor$, is the whole difficulty. Once you set yourself the goal "replace $b$ by $b \bmod a$", each of the remaining steps follows from asking how that goal can be met under the doubling constraint.

**Layer $+\alpha$ BACKTESTING**

**1.** The first useful cue is $y \mapsto y - x$: a Euclidean step. On seeing it, write "aim: make the minimum strictly smaller; target $b \bmod a$". The second cue is $x \mapsto 2x$; write "the subtrahends available are $a, 2a, 4a, \ldots$".

**2.** A descent argument (well-ordering) whose descent step is a Euclidean division carried out through the binary expansion of the quotient.

**3.** Hesitation comes after greedy strategies cycle and the two-number case is seen to fail. It is reinforced away by stopping the search for a good next move and instead fixing the target of a whole round: $b \to b \bmod a$.

**4.** The trap is local greed — always acting on the two smallest — which cycles already on $\{1, 2, 2\}$. A subtler trap is treating the third number as symmetric with the others; it has a specific role (absorbing the binary zeros) and its size must be checked.

**5.** When a process is a known algorithm with one forced distortion (doubling, a fixed increment, an alternation), write down the undistorted algorithm's goal first and then ask how to realise it despite the distortion. Binary expansions appear whenever the distortion is doubling.

**6.** Reusable decision rule: *to show a process on positive integers terminates, do not design moves one at a time; name a monovariant, name the value you want it to drop to, and design a finite batch of moves that achieves exactly that, using any idle quantity to absorb the side effects.*