# DAWN PUTNAM SESSION

Putnam 1987 — Papers A and B

Putnam 120 Archive

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# Putnam 1987 — Paper A

## 1987 A1

**1987 A1.** Curves $A, B, C$ and $D$ are defined in the plane as follows:

$$A = \left\{ (x,y) : x^2 - y^2 = \frac{x}{x^2+y^2} \right\},$$
$$B = \left\{ (x,y) : 2xy + \frac{y}{x^2+y^2} = 3 \right\},$$
$$C = \left\{ (x,y) : x^3 - 3xy^2 + 3y = 1 \right\},$$
$$D = \left\{ (x,y) : 3x^2y - 3x - y^3 = 0 \right\}.$$

Prove that $A \cap B = C \cap D$.

**Fields:** complex numbers, algebra, plane curves **Position:** A1 — the opener **Answer:** Put $z = x + iy$. The pair of equations for $A, B$ is the single complex equation $z^2 - z^{-1} = 3i$; the pair for $C, D$ is $z^3 - 3iz = 1$. These are equivalent for $z \neq 0$ (multiply or divide by $z$), and $z = 0$ lies in neither intersection. Hence $A \cap B = C \cap D$.

### Layer 1 THE WHAT — surface analysis

*The natural attempt.* You read four real equations and set out to show that the solution set of the first pair equals the solution set of the second pair. The natural plan is elimination: clear denominators in $A$ and $B$,

$$(x^2 - y^2)(x^2 + y^2) = x, \qquad 2xy(x^2 + y^2) + y = 3(x^2 + y^2),$$

two quartics, and try to derive $C$ and $D$ from them. You might try to express $x$ from the first ($x = x^4 - y^4$) and substitute into the second, or form combinations like $x \cdot (\text{first}) - y \cdot (\text{second})$ at random. Very quickly the degree climbs to six or eight, and you are holding polynomials in two variables with no idea which combination is meant to collapse to the cubic $x^3 - 3xy^2 + 3y - 1$.

Suppose you are lucky, or patient, and find that multiplying the $A$-equation by $x$ and subtracting $y$ times the $B$-equation gives

$$x(x^2 - y^2) - 2xy^2 - \frac{x^2+y^2}{x^2+y^2} + 3y = x^3 - 3xy^2 + 3y - 1,$$

which is the $C$-expression. Then you have shown $A \cap B \subseteq C$. You still need $D$, and — the part most students drop — you still need the reverse inclusion $C \cap D \subseteq A \cap B$, which requires inverting the combination. Without a reason for the coefficients $x$ and $-y$, you have no systematic way to find the second combination or to invert the pair. The attempt either stalls or produces one direction only.

*The model chain.* For $(x,y) \neq (0,0)$ write $z = x + iy$, so $z \neq 0$ and

$$z^2 = (x^2 - y^2) + 2xy\,i, \qquad \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}\,i.$$

Hence

$$z^2 - \frac{1}{z} = \left(x^2 - y^2 - \frac{x}{x^2+y^2}\right) + \left(2xy + \frac{y}{x^2+y^2}\right)i.$$

A complex number equals $3i$ exactly when its real part is 0 and its imaginary part is 3, so

$$(x,y) \in A \cap B \iff z^2 - \frac{1}{z} = 3i. \tag{1}$$

The defining expressions of $A$ and $B$ are undefined at $(0,0)$, so the origin is in neither $A$ nor $B$; (1) covers every point of $A \cap B$.

Next, $z^3 = (x^3 - 3xy^2) + (3x^2y - y^3)i$ and $-3iz = 3y - 3x\,i$, so

$$z^3 - 3iz - 1 = (x^3 - 3xy^2 + 3y - 1) + (3x^2y - 3x - y^3)\,i,$$

and for every $(x,y)$,

$$(x,y) \in C \cap D \iff z^3 - 3iz = 1. \tag{2}$$

The origin is not in $C$ (it gives $0 = 1$), so every point of $C \cap D$ has $z \neq 0$.

Finally, for $z \neq 0$, multiplying by $z$ and dividing by $z$ are inverse operations, so

$$z^2 - \frac{1}{z} = 3i \iff z^3 - 1 = 3iz \iff z^3 - 3iz = 1.$$

Combining with (1) and (2), a point $(x,y) \neq (0,0)$ lies in $A \cap B$ if and only if it lies in $C \cap D$, and the origin lies in neither. Therefore $A \cap B = C \cap D$. ■

**Layer 2 THE HOW — the bifurcation**

**Bifurcating point.** Both chains start from the same four equations. They split on the first reading, before any algebra. The natural attempt sees four real equations in $x, y$ and reaches for elimination. The model sees that $x^2 - y^2$ and $2xy$ sit side by side, that $x^3 - 3xy^2$ and $3x^2y - y^3$ sit side by side, and that $\frac{x}{x^2+y^2}$, $\frac{-y}{x^2+y^2}$ are the parts of $1/z$ — and pairs the equations into two complex equations. After that the problem is one line: multiply by $z$.

> *$x^2 - y^2$ and $2xy$ — those are the real and imaginary parts of $(x+iy)^2$. And $x^3 - 3xy^2$, $3x^2y - y^3$ are the parts of $(x+iy)^3$. The problem is written in complex numbers with the i taken out. Put it back: set $z = x + iy$ and see what each pair of equations says as one equation.*

> *$\frac{x}{x^2+y^2}$ with $x^2 + y^2$ underneath is $\bar{z}/|z|^2$, which is $1/z$. So $A \cap B$ is $z^2 - 1/z = 3i$, and $C \cap D$ is $z^3 - 3iz = 1$. One is the other times $z$. The only thing to police is $z = 0$.*

**Layer 3 THE WHY**

Why is going complex not a trick? Start from what the complex numbers *are* as a structure on the plane: $\mathbb{C}$ is $\mathbb{R}^2$ with a multiplication, and the definition of that multiplication, $(x+iy)(u+iv) = (xu - yv) + (xv + yu)i$, is exactly the rule that produces paired expressions of the shape "$x^2 - y^2$, $2xy$". Every polynomial in $z$, split into real and imaginary parts, produces a *pair* of real polynomials, and those pairs have a fingerprint: the binomial expansion of $(x+iy)^n$ with alternating signs, $x^n - \binom{n}{2}x^{n-2}y^2 + \cdots$ and $nx^{n-1}y - \binom{n}{3}x^{n-3}y^3 + \cdots$. When a problem hands you the fingerprint — a 3 and a $-3$ in exactly the binomial places — it has told you where the expressions came from. Recognising it is recognising a definition, in the same way that seeing $(x-a)^2 + (y-b)^2 = r^2$ is recognising a circle.

There is also a structural reason the elimination route is doomed to be painful. Two real equations defining a set of points is one complex equation if and only if the two equations are

the real and imaginary parts of the same complex function. Real elimination does not know this; it treats the equations as unrelated and pays for the ignorance in degree. In complex form, $z^2 - 1/z = 3i$ is a cubic in one unknown, $z^3 - 3iz - 1 = 0$, with at most three roots. So the whole intersection $A \cap B$ is at most three points. A four-variable-looking elimination problem is secretly "solve a cubic", and nothing about the real formulation shows you that.

Now look at the official Solution 2 in Kedlaya–Poonen–Vakil. It writes the matrix identity

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} H \\ J \end{pmatrix},$$

where $F, G, H, J$ are the four defining expressions (with $G$ including the $-3$). Here is where that matrix comes from: $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ is the matrix of "multiply by $z = x + iy$" acting on $\mathbb{R}^2 = \mathbb{C}$. It is the real-coordinate shadow of the one-line complex step. The coefficients $x$ and $-y$ that the natural attempt found by luck are the first row of that matrix; the second row, $(y, x)$, is the combination that gives $D$; and the determinant $x^2 + y^2 = |z|^2$ is why the step reverses exactly when $z \neq 0$. So the official Solution 2 is not a different idea. It is the same idea with the reason removed. The book's own remark says as much.

Finally, the rigour point is not decoration. "Multiply both sides by $z$" is an equivalence only when $z \neq 0$. The problem has arranged that the origin is harmless: it is excluded from $A$ and $B$ because their equations are undefined there, and from $C$ because $0 \neq 1$. But you must say so, because the equivalence you are using is only a one-way implication at $z = 0$.

### Layer $+\alpha$ BACKTESTING

**1.** The first useful cue is the pair $x^2 - y^2$, $2xy$ in the definitions of $A$ and $B$. On seeing it, write $z = x + iy$ and $z^2 = (x^2 - y^2) + 2xyi$ in the margin before doing anything else.

**2.** It suggests rewriting each pair of real equations as one complex equation, then comparing the two complex equations directly.

**3.** Hesitation comes at $\frac{x}{x^2+y^2}$ and $\frac{y}{x^2+y^2}$: they do not look like powers of $z$. Reinforce it away with the identity $1/z = \bar{z}/|z|^2$, which turns any $\frac{(\text{linear in } x,y)}{x^2+y^2}$ into a multiple of $1/z$ or $1/\bar{z}$. Watch the sign: $\operatorname{Im}(1/z) = -y/(x^2+y^2)$, so "$+\frac{y}{x^2+y^2}$" is $-\operatorname{Im}(1/z)$.

**4.** Two traps. First, real elimination, which rises to high degree and gives at best one inclusion. Second, proving only $A \cap B \subseteq C \cap D$, or dividing by $z$ without checking that $z \neq 0$ on $C \cap D$.

**5.** Train the fingerprint: $(x^2 - y^2,\, 2xy)$, $(x^3 - 3xy^2,\, 3x^2y - y^3)$, $\frac{(x,\,-y)}{x^2+y^2}$. Whenever two real equations share these shapes, they are one complex equation. The same goes for a $2 \times 2$ matrix of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$: it is multiplication by $a + bi$.

**6.** Reusable rule: before eliminating between two real equations in $x, y$, ask whether they are the real and imaginary parts of one complex equation in $z = x + iy$. If they are, solve or transform the complex equation, and convert back only at the end. Every multiplication or division you perform must be checked for the value zero.

▷ **Cue:** two real equations in $x, y$ containing $x^2 - y^2$ and $2xy$ (or $x^3 - 3xy^2$ and $3x^2y - y^3$) in matching positions $\longrightarrow$ **Move:** set $z = x + iy$ and merge the pair into one complex equation

▷ **Cue:** terms $\frac{x}{x^2+y^2}$ and $\frac{y}{x^2+y^2}$ $\longrightarrow$ **Move:** recognise $1/z = \bar{z}/|z|^2 = \frac{x-iy}{x^2+y^2}$

▷ **Cue:** a $2 \times 2$ real matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ linking two pairs of equations $\longrightarrow$ **Move:** it is multiplication by $a + bi$; its determinant $a^2 + b^2$ tells you when the step is reversible

▷ **Cue:** an equivalence obtained by multiplying through by a variable quantity $\longrightarrow$ **Move:** check separately the points where that quantity is zero, in both directions

## 1987 A2

**1987 A2.** The sequence of digits

$$123456789101112131415161718192021\ldots$$

is obtained by writing the positive integers in order. If the $10^n$-th digit in this sequence occurs in the part of the sequence in which the $m$-digit numbers are placed, define $f(n)$ to be $m$. For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

**Fields:** counting, estimation, geometric sums **Position:** A2 **Answer:** $f(1987) = 1984$. If $g(m)$ is the number of digits used by all integers with at most $m$ digits, then $g(1983) < 1983 \cdot 10^{1983} < 10^{1987}$ and $g(1984) > 1983 \cdot 10^{1984} > 10^{1987}$.

### Layer 1 THE WHAT — surface analysis

*The natural attempt.* You start the right way: count digits block by block. There are 9 one-digit numbers, 90 two-digit numbers, 900 three-digit numbers, and in general $9 \cdot 10^{r-1}$ numbers with $r$ digits, contributing $9r \cdot 10^{r-1}$ digits. So the number of digits used up to the end of the $m$-digit block is

$$g(m) = \sum_{r=1}^{m} 9r \cdot 10^{r-1},$$

and $f(n)$ is the $m$ with $g(m-1) < 10^n \le g(m)$. (Check: $g(1) = 9 < 100 \le g(2) = 189$, so $f(2) = 2$.)

Then you go looking for $m$. The strong student now does one of two things. Either they set out to find a closed form for $\sum r \cdot 10^{r-1}$ — differentiate a geometric series, or use the shift-and-subtract trick — and arrive, after a page and some sign errors, at $g(m) = m \cdot 10^m - \frac{10^m-1}{9}$, and then try to *solve* $m \cdot 10^m - \frac{10^m-1}{9} = 10^{1987}$ for $m$. Or they take logarithms: $g(m) \approx m \cdot 10^m$, so $m + \log_{10} m \approx 1987$, and since $\log_{10} m$ is about 3.3, $m \approx 1983.7$, so "$m = 1984$".

The first route produces an equation that cannot be solved exactly (it mixes $m$ with $10^m$), and the student stares at it. The second route gets the right number and proves nothing: every "$\approx$" is unjustified, and the answer sits between 1983 and 1984 with a margin of 0.3 in the logarithm, which is exactly the kind of margin where a careless approximation flips the answer. Many scripts end here with the right answer and a proof mark of zero.

*The model chain.* Let $g(m)$ denote the total number of digits in all positive integers with at most $m$ digits, and $g(0) = 0$. The $m$-digit integers occupy positions $g(m-1)+1, \ldots, g(m)$ of the sequence, so by definition $f(n)$ is the unique $m \ge 1$ with

$$g(m-1) < 10^n \le g(m).$$

There are exactly $10^r - 10^{r-1} = 9 \cdot 10^{r-1}$ integers with $r$ digits, so

$$g(m) = \sum_{r=1}^{m} r\,(10^r - 10^{r-1}).$$

*Upper bound for $g(1983)$.* Each $r \le 1983$, so

$$g(1983) \le 1983 \sum_{r=1}^{1983} (10^r - 10^{r-1}) = 1983\,(10^{1983} - 1) < 1983 \cdot 10^{1983} < 10^4 \cdot 10^{1983} = 10^{1987},$$

the sum telescoping.

*Lower bound for* $g(1984)$. Keep only the last term:

$$g(1984) \geq 1984\,(10^{1984} - 10^{1983}) = 1984 \cdot 9 \cdot 10^{1983} = 17856 \cdot 10^{1983} > 10^4 \cdot 10^{1983} = 10^{1987}.$$

Hence $g(1983) < 10^{1987} < g(1984)$, and $f(1987) = 1984$. ■

**Layer 2 THE HOW — the bifurcation**

**Bifurcating point.** Both chains reach $g(m) = \sum_{r \leq m} 9r \cdot 10^{r-1}$ and the condition $g(m-1) < 10^{1987} \leq g(m)$. They split on the next line. The natural attempt tries to *compute* $g$ (closed form, then solve, or logarithms, then round). The model only *brackets* it: one crude upper bound at $m = 1983$ (replace every $r$ by 1983 and telescope), one crude lower bound at $m = 1984$ (throw away every term but the last). Both bounds miss by a factor of about 5 and 2 respectively and still win, because the question only asks which side of $10^{1987}$ each value falls.

> *I don't need $g(m)$; I need to know which side of $10^{1987}$ it lies on for two consecutive values of $m$. That is two inequalities, not an equation. Let me guess $m$ first, then prove the guess with the crudest bounds that still work.*

> *The sum is dominated by its last term, $9m \cdot 10^{m-1}$, because it is geometric-like with ratio 10. So $g(m)$ is roughly $m \cdot 10^m$, and $m \cdot 10^m = 10^{1987}$ wants $m$ with four digits, around $10^{1987-m}$: $m = 1984$ gives $1984 \approx 10^3$. Guess 1984; check 1983 and 1984.*

**Layer 3 THE WHY**

Why is bracketing the natural move and not a shortcut you had to be shown? Go back to what is actually being asked. $f(n)$ is defined as "the block in which the $10^n$-th digit falls". A block is an interval of positions, $(g(m-1), g(m)]$. Membership of an interval is, by definition, two inequalities. The problem never asks for the value of $g$ anywhere; it asks for a comparison. An exact closed form is information you will immediately throw away, so computing it first is zombie maths in its purest form — effort spent on precision that the question does not reward.

Then why are *crude* bounds enough? Because the numbers live on a scale where the relevant quantities differ by whole factors of 10. Moving $m$ by one multiplies the dominant term $m \cdot 10^m$ by roughly 10. Any error in estimating $g$ that is smaller than a factor of 10 therefore cannot move the answer by more than one, and here the estimates were chosen so the errors land on the correct side. That is the structural fact to carry away: in a sum where each term is about ten times the previous one, the last term is the sum up to a constant factor — precisely, $9m \cdot 10^{m-1} \leq g(m) \leq m \cdot 10^m$. The upper bound came from replacing every coefficient $r$ by its largest value and telescoping $\sum(10^r - 10^{r-1})$, the lower from discarding everything but the last term. Both are the first thing you would try; neither needs cleverness.

Where does the guess 1984 come from? The official solution's "Motivation" says it: set $m \cdot 10^m \approx 10^{1987}$, so $m \approx 10^{1987-m}$. Since $m$ is near 2000, $10^{1987-m}$ must be near 2000, so $1987 - m$ is 3 (as $10^3 \leq m < 10^4$), giving $m = 1984$. Note what this reasoning does and does not establish: it makes 1983 and 1984 the candidates, and it is honest to admit the logarithm margin is thin ($1983 + \log_{10} 1983 \approx 1986.3$, $1984 + \log_{10} 1984 \approx 1987.3$). That thinness is why the proof must be inequalities with room to spare, not rounded logarithms. The date-flavoured answer (1984 for a 1987 paper) is also a hint that the answer is $n - 3$; that is not evidence, but it is a reason to check your arithmetic if you get something else.

The closed form $g(m) = m \cdot 10^m - \frac{10^m - 1}{9}$ that the book gives as a remark is worth knowing (it comes from $\sum_r P(r)x^r$ by shift-and-subtract, reducing deg $P$ by one each time), and it gives a second proof. But notice the book's main proof does not use it. The authors, who knew the closed form, chose the two-line bracket. That choice is the lesson.

**Layer $+\alpha$ BACKTESTING**

**1.** The first cue is "the $10^n$-th digit occurs in the part … in which the $m$-digit numbers are placed". Write down: $g(m) =$ digits used through the $m$-digit block, and $f(n) = m \iff g(m-1) < 10^n \leq g(m)$.

**2.** It suggests counting by blocks ($9 \cdot 10^{r-1}$ numbers of $r$ digits) and then locating $10^{1987}$ between two consecutive values of $g$ by inequalities.

**3.** Hesitation occurs when the equation $g(m) = 10^{1987}$ refuses to be solved. Reinforce it away by re-reading the definition: an interval condition is two inequalities, and inequalities tolerate crude estimates.

**4.** The trap is the heuristic $m + \log_{10} m \approx 1987$, which gives the right answer and no proof, with a margin thin enough that a sloppy rounding flips it. The secondary trap is spending the time on the exact closed form.

**5.** Any time the answer is "which block/range/interval does a huge index fall into", the task is to bracket, not to compute. Look for the dominant term of the cumulative count and bound the rest by it.

**6.** Reusable rule: when a sum grows geometrically, its last term controls it to within a constant factor; bound above by (number of terms or largest coefficient) $\times$ (telescoped geometric part) and below by the last term alone. Guess the answer by the heuristic, then prove it with the two crudest bounds that separate the neighbouring candidates.

▷ **Cue:** "in which block / range does the $N$-th item fall" with $N$ astronomically large $\longrightarrow$ **Move:** define the cumulative count $g(m)$ and reduce to $g(m-1) < N \leq g(m)$

▷ **Cue:** a cumulative count $\sum_{r \leq m} c_r b^r$ with $c_r$ polynomial and base $b \geq 2$ $\longrightarrow$ **Move:** the last term dominates: bound below by it, above by $\max c_r$ times the telescoped geometric sum

▷ **Cue:** an equation mixing $m$ and $10^m$ that cannot be solved exactly $\longrightarrow$ **Move:** take logarithms only to guess, then verify the guess and its neighbour with explicit inequalities

▷ **Cue:** a sum $\sum_{r=a}^{b} P(r)x^r$ that you genuinely need in closed form $\longrightarrow$ **Move:** multiply by $(1-x)$ and shift, lowering $\deg P$ by one each time (or differentiate the geometric series)

## 1987 A3

**1987 A3.** For all real $x$, the real-valued function $y = f(x)$ satisfies

$$y'' - 2y' + y = 2e^x.$$

(a) If $f(x) > 0$ for all real $x$, must $f'(x) > 0$ for all real $x$? Explain.

(b) If $f'(x) > 0$ for all real $x$, must $f(x) > 0$ for all real $x$? Explain.

**Fields:** differential equations, quadratics **Position:** A3 **Answer:** (a) No: $f(x) = (x^2+1)e^x$ is positive everywhere but $f'(x) = (x+1)^2 e^x$ vanishes at $x = -1$. (b) Yes. Every solution is $f = (x^2 + bx + c)e^x$; with $u = x^2 + bx + c$ one has $f'(x) = \big(u(x+1) - 1\big)e^x$, so $f' > 0$ everywhere means $\min u > 1$, which forces $f > 0$.

**Layer 1 THE WHAT — surface analysis**

*The natural attempt.* A student with Further Maths does what the syllabus trained: complementary function plus particular integral. The auxiliary equation $r^2 - 2r + 1 = 0$ has the repeated root $r = 1$, so the complementary function is $(bx + c)e^x$. For the particular integral you try $ke^x$ — it gives 0, because $e^x$ is in the complementary function. You try $kxe^x$ — also 0. You remember the rule for a repeated root and try $kx^2e^x$: after two differentiations, $k(x^2 + 4x + 2)e^x - 2k(x^2 + 2x)e^x + kx^2e^x = 2ke^x$, so $k = 1$. General solution $f = (x^2 + bx + c)e^x$. That took several lines and two false starts, but it works.

Then you differentiate: $f'(x) = \big(x^2 + (b+2)x + b + c\big)e^x$. Now you must decide whether "$x^2 + bx + c > 0$ for all $x$" forces "$x^2 + (b+2)x + b + c > 0$ for all $x$" and conversely. Here the natural attempt often goes case by case: try $b = 0$, try $c$ large, sketch parabolas, try to argue from the vertex of one parabola to the other. For (a) you may even try to *prove* yes ("$f$ is positive and grows like $x^2e^x$, so it should be increasing") before noticing that for $x \to -\infty$, $f$ decays to 0 and nothing forces monotonicity. The pieces are all there, but the comparison between two parabolas with two parameters is done by feel, not by a single inequality.

*The model chain.* *Solving.* Write $f(x) = u(x)e^x$, which defines a twice-differentiable $u = fe^{-x}$. Then $f' = (u' + u)e^x$ and $f'' = (u'' + 2u' + u)e^x$, so

$$f'' - 2f' + f = \big(u'' + 2u' + u - 2u' - 2u + u\big)e^x = u''e^x.$$

The equation becomes $u''e^x = 2e^x$, i.e. $u'' = 2$ for all real $x$. Integrating twice, $u(x) = x^2 + bx + c$ for some real constants $b, c$, and conversely every such $u$ gives a solution. So the solutions are exactly $f(x) = (x^2 + bx + c)e^x$.

*The two conditions.* Since $e^x > 0$, the sign of $f$ is the sign of $u$ and the sign of $f'$ is the sign of $u' + u$. Let $\Delta = b^2 - 4c$. A monic quadratic is positive for every real $x$ if and only if its discriminant is negative (if the discriminant is $\geq 0$ it has a real root, where it is 0). So

$$f(x) > 0 \text{ for all } x \iff \Delta < 0.$$

Now $u' + u = x^2 + (b+2)x + (b+c)$, with discriminant

$$(b+2)^2 - 4(b+c) = b^2 - 4c + 4 = \Delta + 4,$$

so

$$f'(x) > 0 \text{ for all } x \iff \Delta + 4 < 0 \iff \Delta < -4.$$

*(a)* No. $\Delta < 0$ does not imply $\Delta < -4$. Take $b = 0$, $c = 1$: $f(x) = (x^2 + 1)e^x > 0$ for all $x$, but $f'(x) = (x^2 + 2x + 1)e^x = (x+1)^2e^x$, and $f'(-1) = 0$. (With $b = c = 1$ one even gets $f' < 0$ on $(-2, -1)$, since $u' + u = (x+1)(x+2)$.)

*(b)* Yes. If $f' > 0$ for all $x$ then $\Delta < -4 < 0$, so $f > 0$ for all $x$. ■

**Layer 2 THE HOW — the bifurcation**

**Bifurcating point.** There are two forks. The first is at "solve the equation": the natural attempt runs the CF+PI algorithm and fumbles through resonance; the model divides out $e^x$ and gets $u'' = 2$ in one line. The second, more important, is after $f' = \big(x^2 + (b+2)x + b + c\big)e^x$ is on the page. The natural attempt compares two parabolas by sketching and cases; the model writes both positivity conditions as discriminant conditions and subtracts: $\Delta' = \Delta + 4$. Once that one identity is written, (a) and (b) are both read off.

> *The right-hand side is $e^x$ and the operator has the double root 1 — everything is $e^x$ times something. Let me write $f = ue^x$ and see what the equation says about $u$. It says $u'' = 2$. So $u$ is a monic quadratic; no guessing of particular integrals.*

> *$e^x$ is positive, so "$f > 0$ everywhere" and "$f' > 0$ everywhere" are statements about two monic quadratics being positive everywhere. That is a discriminant condition each time. Compute both discriminants and compare them; don't draw parabolas.*

**Layer 3 THE WHY**

Why is $f = ue^x$ the natural first move? Look at the operator: $y'' - 2y' + y = (D-1)^2y$, where $D$ is differentiation. The structural fact is the "shift rule": $(D-1)(ue^x) = u'e^x$, which is the

product rule and nothing else — $(ue^x)' - ue^x = u'e^x$. Apply it twice and $(D-1)^2(ue^x) = u''e^x$. So multiplying by $e^x$ turns the awkward operator $(D-1)^2$ into the plainest operator there is, $D^2$. The problem announces this twice: the characteristic root is 1 (double), and the forcing term is $e^x$. When the forcing matches the root, the CF+PI algorithm calls it "resonance" and makes you guess $x^2e^x$; the substitution shows you why $x^2$ appears — it is what you get by integrating the constant 2 twice.

Why is the discriminant comparison natural rather than lucky? The question asks about positivity *for all real* $x$. For a quadratic with positive leading coefficient, positivity everywhere is, by definition of "no real root" plus continuity, exactly the discriminant condition. So the problem is asking you to compare two discriminants, and the difference $\Delta' - \Delta = 4$ is a constant independent of $b$ and $c$ — which is the reason the answer is an asymmetric "no, yes". One condition is strictly stronger than the other by a fixed margin.

There is a cleaner way to see that margin, and it is worth having because it explains the 4. Since $u'' = 2$, Taylor's formula for a quadratic is exact:

$$u(x+1) = u(x) + u'(x) + \tfrac{1}{2}u'' = u(x) + u'(x) + 1.$$

So $u' + u = u(x+1) - 1$, and

$$f'(x) = \big(u(x+1) - 1\big)e^x.$$

Now the conditions are transparent. Writing $\mu = \min_{\mathbb{R}} u = c - \frac{b^2}{4} = -\frac{\Delta}{4}$: $f > 0$ everywhere means $\mu > 0$; $f' > 0$ everywhere means $u(x+1) > 1$ for all $x$, i.e. $\mu > 1$. And $\mu > 1$ implies $\mu > 0$, not conversely. The counterexample in (a) is any $u$ with $0 < \mu \le 1$, such as $u = x^2 + 1$ with $\mu = 1$ exactly, where $f'$ touches zero at $x = -1$ (the point with $x + 1$ at the vertex). The "+4" in the discriminant is the "−1" in $u(x+1) - 1$, scaled by $-4$.

Kedlaya–Poonen–Vakil give the discriminant version and say "Clearly $b^2 - 4c < 0$ does not imply $b^2 - 4c + 4 < 0$". The step owes a reason and the reason is above: $f' > 0$ asks the quadratic to clear height 1, $f > 0$ only asks it to clear height 0.

Finally, why must (a) be "no" on qualitative grounds, before any computation? Every solution behaves like $x^2e^x \to 0$ as $x \to -\infty$. A positive function tending to 0 at $-\infty$ must be increasing "on average" there, but nothing prevents a flat spot or a dip. The problem-setter's (a) is testing whether you will confuse "positive and eventually growing" with "increasing".

### Layer $+\alpha$ BACKTESTING

**1.** The first cue is the operator $y'' - 2y' + y$ (double root 1) together with the forcing $2e^x$. Write $f = ue^x$ and reduce to $u'' = 2$ at once.

**2.** It suggests the substitution $y = ue^{rx}$ for an operator $(D-r)^2$, then turning both questions into sign questions about explicit quadratics, settled by discriminants.

**3.** Hesitation occurs when comparing the two parabolas $u$ and $u' + u$. Reinforce it away by computing both discriminants and subtracting, or by the identity $u' + u = u(x+1) - 1$.

**4.** Trap one: guessing $ke^x$ and $kxe^x$ for the particular integral and losing time. Trap two: the plausible belief that a positive solution growing like $x^2e^x$ must be increasing. Trap three: stating the positivity criterion as $\Delta \le 0$, which admits a zero.

**5.** Any linear ODE whose characteristic root matches the exponential on the right-hand side is begging for $y = ue^{rx}$. Any "for all real $x$" positivity question about a quadratic, or a quadratic times a positive function, is a discriminant question.

**6.** Reusable rule: factor out the known positive part (here $e^x$) so that sign questions become questions about a simpler function; then express each "for all $x$" condition as a single inequality on the parameters and compare the inequalities directly. An implication between conditions is an inclusion between parameter regions.

▷ **Cue:** a constant-coefficient linear ODE with repeated root $r$ and forcing $e^{rx}$ ⟶ **Move:** write $y = ue^{rx}$; the operator $(D-r)^2$ becomes $D^2$, so $u''$ equals the forcing coefficient

- ▷ **Cue:** a question about the sign of $p(x)e^{g(x)}$ for all $x$ $\longrightarrow$ **Move:** discard the positive factor; ask only about the sign of $p$
- ▷ **Cue:** "positive for all real $x$" for a quadratic with positive leading coefficient $\longrightarrow$ **Move:** equivalent to discriminant $< 0$ (strict), or to the minimum value $> 0$
- ▷ **Cue:** two positivity conditions depending on the same parameters, and the question "does one imply the other" $\longrightarrow$ **Move:** reduce each to an inequality in the parameters and compare the regions; produce a counterexample on the boundary
- ▷ **Cue:** $u + u'$ with $u$ a quadratic of leading coefficient $a$ $\longrightarrow$ **Move:** $u(x) + u'(x) = u(x+1) - a$, by exact Taylor expansion

## 1987 A4

**1987 A4.** Let $P$ be a polynomial, with real coefficients, in three variables and $F$ be a function of two variables such that

$$P(ux, uy, uz) = u^2 F(y - x, z - x) \quad \text{for all real } x, y, z, u,$$

and such that $P(1,0,0) = 4$, $P(0,1,0) = 5$, and $P(0,0,1) = 6$. Also let $A, B, C$ be complex numbers with $P(A,B,C) = 0$ and $|B - A| = 10$. Find $|C - A|$.

**Fields:** algebra (homogeneous polynomials), complex numbers **Position:** A4 — mid-paper, but in truth one of the gentler problems of 1987 once the functional equation is read as two separate facts **Answer:** $|C - A| = \frac{5}{3}\sqrt{30}$. The hypothesis forces $P(x,y,z) = 5(y-x)^2 - 7(y-x)(z-x) + 6(z-x)^2$; with $m = (C-A)/(B-A)$ this gives $6m^2 - 7m + 5 = 0$, whose roots are complex conjugates with product $\frac{5}{6}$, so $|m| = \sqrt{5/6}$ and $|C - A| = 10\sqrt{5/6}$.

### Layer 1 THE WHAT — surface analysis

*The natural attempt.* You see a polynomial in three variables and three numerical values, and your first instinct is to write $P$ out in general. A polynomial of degree at most 2 in $x, y, z$ has ten coefficients: $1, x, y, z, x^2, y^2, z^2, xy, yz, zx$. You have three equations. You try to squeeze more out of the identity by plugging in $u = 2$, or $x = y = z$, and each substitution gives a relation, but you are pushing numbers round a ten-unknown system without a picture of what the identity is telling you.

A second natural route is to take the identity at face value with $u$ fixed at 1, getting $P(x,y,z) = F(y - x, z - x)$, and then try to use the three values of $P$ to pin down $F$. But $F$ is "a function of two variables" — nothing says polynomial — and three values of an arbitrary function tell you nothing. You stall: either the system is underdetermined, or $F$ is too unconstrained.

Even if you do reach the quadratic form, the final step can trap you. You have $5\beta^2 - 7\beta\gamma + 6\gamma^2 = 0$ with $\beta = B - A$, $\gamma = C - A$ complex, and a student used to real numbers may try to solve for $\gamma$ with the quadratic formula, meet $\sqrt{-71}$, and wonder whether there is any solution at all — or compute $|\gamma|$ by expanding real and imaginary parts, four real unknowns and a page of algebra.

*The model chain.* *Step 1: $F$ is a polynomial.* Put $u = 1$ and $x = 0$ in the identity: $F(y,z) = P(0,y,z)$ for all real $y, z$. So $F$ is a polynomial with real coefficients.

*Step 2: $P$ is invariant under translation along $(1,1,1)$.* Put $u = 1$: $P(x,y,z) = F(y - x, z - x)$ for all real $x, y, z$.

*Step 3: $F$ is homogeneous of degree 2.* Put $x = 0$ in the identity: $P(0, uy, uz) = u^2 F(y,z)$, and by Step 1 the left side is $F(uy, uz)$. So $F(uy, uz) = u^2 F(y,z)$ for all real $u, y, z$. Write $F = \sum_d F_d$ with $F_d$ homogeneous of degree $d$; then $\sum_d u^d F_d(y,z) = u^2 \sum_d F_d(y,z)$ as polynomials in $u$ for each

fixed $(y,z)$, so $F_d(y,z)=0$ for every $d\neq 2$ and every real $(y,z)$, hence $F_d$ is the zero polynomial for $d\neq 2$. Therefore

$$F(s,t)=as^2+bst+ct^2,\qquad a,b,c\in\mathbb{R}.$$

*Step 4: the constants.* By Step 2, $P(0,1,0)=F(1,0)=a=5$, $P(0,0,1)=F(0,1)=c=6$, $P(1,0,0)=F(-1,-1)=a+b+c=4$, so $b=-7$ and

$$P(x,y,z)=5(y-x)^2-7(y-x)(z-x)+6(z-x)^2.$$

This is an identity of polynomials that holds on all of $\mathbb{R}^3$, so the two sides have the same coefficients and it holds for complex $x,y,z$ as well.

*Step 5: the complex condition.* Let $\beta=B-A$ and $\gamma=C-A$. Then $0=P(A,B,C)=5\beta^2-7\beta\gamma+6\gamma^2$. Since $|\beta|=10$, $\beta\neq 0$; divide by $\beta^2$ and set $m=\gamma/\beta$:

$$6m^2-7m+5=0.$$

The discriminant is $49-120=-71<0$, so the two roots are non-real complex conjugates $m,\bar{m}$ (real coefficients). Their product is $m\bar{m}=|m|^2=\frac{5}{6}$ by Vieta. Whichever root $m$ is,

$$|C-A|=|m|\,|B-A|=10\sqrt{\tfrac{5}{6}}=\tfrac{10\sqrt{30}}{6}=\tfrac{5}{3}\sqrt{30}.$$

Both roots give the same modulus, so the answer is determined.

**Layer 2 THE HOW — the bifurcation**

**Bifurcating point.** Both chains start at the displayed identity. The natural attempt treats it as one equation to be fed numbers. The model splits it at once into its two independent pieces — set $u=1$ (translation invariance) and set $x=0$ (scaling) — and at the end divides by $\beta^2$ rather than solving for $\gamma$.

> *The identity has two free ingredients, the scale $u$ and the position $x,y,z$. I'll switch them off one at a time. With $u=1$, $P$ only sees differences $y-x$ and $z-x$. With $x=0$, $F$ scales like $u^2$, so $F$ is a quadratic form in two variables: three coefficients, three data. That is why I was given exactly three values.*

> *I am asked for a modulus, not a number. The equation $5\beta^2-7\beta\gamma+6\gamma^2=0$ is homogeneous, so only the ratio $\gamma/\beta$ matters; a real quadratic with negative discriminant has conjugate roots, and the modulus of a conjugate pair is the square root of the product.*

**Layer 3 THE WHY**

Why is splitting the identity the natural move rather than a trick? Look at what the identity *is*. The left side depends on $(u,x,y,z)$; the right side is built from $u^2$ and two differences. That shape says exactly two things about $P$: it does not care about adding the same amount to all three variables (because only $y-x$ and $z-x$ survive), and it scales by $u^2$ when you scale the inputs by $u$. Any identity that mixes a parameter with the variables is a bundle of structural statements, and the way to unbundle it is to fix each ingredient at its neutral value — $u=1$ is the identity scaling, $x=0$ is the base point. Nothing more is going on.

Why does homogeneity reduce the problem to three unknowns? A polynomial is a sum of homogeneous pieces, and scaling multiplies the degree-$d$ piece by $u^d$. If the whole thing scales by $u^2$, the pieces of every other degree must be zero, because distinct powers of $u$ are linearly independent as functions of $u$. In two variables the degree-2 monomials are $s^2,st,t^2$ — three of them. The problem gives three values. The count matching is not a coincidence you stumble on; it is the setter telling you that you have correctly understood the structure. When a problem's

data count equals the dimension of the space you have reduced to, you are on the intended path.

Why is Step 1 needed at all? Because the problem is careful to say $F$ is only "a function". Without knowing $F$ is a polynomial you cannot decompose it into homogeneous parts. The model extracts polynomiality for free from $F(y,z) = P(0,y,z)$. Kedlaya–Poonen–Vakil's solution opens with exactly this line, "letting $u = 1$ and $x = 0$", and does not say why: here is where it comes from. It is the only substitution that isolates $F$ on one side and something known on the other.

Why the ratio? The final equation $5\beta^2 - 7\beta\gamma + 6\gamma^2 = 0$ is homogeneous of degree 2 in $(\beta, \gamma)$ — inherited, of course, from the degree-2 homogeneity of $P$. Homogeneous equations never determine their variables, only ratios, so the unknown that the equation actually pins down is $m = \gamma/\beta$. The question asks for $|\gamma|$ given $|\beta|$, which is $|m| \cdot |\beta|$: the problem is asking for the modulus of the ratio in disguise.

Why conjugates? A quadratic with real coefficients and negative discriminant has roots $m$ and $\bar{m}$, and Vieta gives $m\bar{m} = c/a$. But $m\bar{m} = |m|^2$ by the definition of modulus. So the product of the roots is the squared modulus of either one, and you never need the roots themselves. This is the reason you should not reach for the quadratic formula: the formula gives you $m = (7 \pm i\sqrt{71})/12$, and then $|m|^2 = (49 + 71)/144 = 120/144 = 5/6$ — the same number, reached by the long way. Cheapness here is reading the question ("modulus") before computing.

### Layer $+\alpha$ BACKTESTING

**1.** The first useful cue is the functional identity with a free scaling parameter $u$ and differences $y - x$, $z - x$. On seeing it, write down both specialisations: $u = 1$ gives $P(x,y,z) = F(y-x, z-x)$; $x = 0$ gives $F(uy, uz) = u^2 F(y,z)$.

**2.** Homogeneous-decomposition: a polynomial satisfying $F(u\mathbf{v}) = u^2 F(\mathbf{v})$ is a quadratic form, $as^2 + bst + ct^2$. Then use the three data points to find $a, b, c$.

**3.** Hesitation occurs at "$F$ is only a function, not a polynomial" and at "the quadratic in $m$ has no real roots". Reinforce the first with $F(y,z) = P(0,y,z)$; reinforce the second with: complex roots of a real quadratic come in conjugate pairs, and the product of a conjugate pair is the squared modulus.

**4.** The trap is writing out a general polynomial with ten coefficients. A second trap is evaluating $P(1,0,0)$ as $F(1,0)$ instead of $F(0-1, 0-1) = F(-1,-1)$; that sign slip gives $b = -7$ the wrong way and changes the answer.

**5.** Whenever an identity carries a parameter multiplying every variable, read it as a homogeneity statement; whenever it contains only differences of the variables, read it as translation invariance. When an equation in two complex unknowns is homogeneous, divide through to get an equation in their ratio.

**6.** Reusable rule: unbundle a parametrised identity by setting each parameter to its neutral value in turn, and count the dimension of what is left; if that matches the number of data given, you have found the intended structure. For moduli of roots of real polynomials, use Vieta on conjugate pairs rather than solving.

▷ **Cue:** an identity $P(u\mathbf{x}) = u^k(\ldots)$ holding for all real $u$ $\longrightarrow$ **Move:** $P$ is homogeneous of degree $k$; only the degree-$k$ monomials survive

▷ **Cue:** a function appears only through differences $y - x$, $z - x$ $\longrightarrow$ **Move:** translation invariance along $(1, 1, \ldots, 1)$; express everything in the differences

▷ **Cue:** the problem says "$F$ is a function" without saying polynomial $\longrightarrow$ **Move:** find a specialisation writing $F$ as the known polynomial, e.g. $F(y,z) = P(0,y,z)$

▷ **Cue:** the number of given values equals the number of free coefficients after reduction $\longrightarrow$ **Move:** you have reduced correctly; solve the linear system

▷ **Cue:** a homogeneous equation $a\beta^2 + b\beta\gamma + c\gamma^2 = 0$ and a question about $|\gamma|/|\beta|$ $\longrightarrow$ **Move:** set $m = \gamma/\beta$ and work with $am^2 + bm + c = 0$

▷ **Cue:** a real quadratic with negative discriminant and a question about the modulus of a root
⟶ **Move:** $|m|^2 = m\bar{m} = c/a$ by Vieta; never solve for $m$

## 1987 A5

**1987 A5.** Let

$$\vec{G}(x,y) = \left(\frac{-y}{x^2+4y^2}, \frac{x}{x^2+4y^2}, 0\right).$$

Prove or disprove that there is a vector-valued function

$$\vec{F}(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))$$

with the following properties:

(i) $M, N, P$ have continuous partial derivatives for all $(x,y,z) \neq (0,0,0)$;

(ii) $\operatorname{Curl} \vec{F} = \vec{0}$ for all $(x,y,z) \neq (0,0,0)$;

(iii) $\vec{F}(x,y,0) = \vec{G}(x,y)$.

**Fields:** vector calculus, topology of domains **Position:** A5 — hard for its year (only ten of roughly two hundred top scorers made real progress), because the decisive idea is a global one while every local check says "yes" **Answer:** No such $\vec{F}$ exists. If it did, Stokes' theorem on the upper half of the ellipsoid $x^2+4y^2+z^2=4$ (a surface avoiding the origin, bounded by the ellipse $x^2+4y^2=4$, $z=0$) would give $\oint \vec{G}\cdot d\vec{r} = 0$; but that circulation equals $\pi$.

### Layer 1 THE WHAT — surface analysis

*The natural attempt.* You check the obvious necessary condition first. In the plane, a curl-free field has $\partial_x G_2 - \partial_y G_1 = 0$. With $D = x^2+4y^2$,

$$\partial_x\left(\frac{x}{D}\right) = \frac{D-2x^2}{D^2} = \frac{4y^2-x^2}{D^2}, \qquad \partial_y\left(\frac{-y}{D}\right) = -\frac{D-8y^2}{D^2} = \frac{4y^2-x^2}{D^2},$$

so the $z$-component of $\operatorname{Curl}\vec{G}$ is 0. The other two components vanish because $G$ has no $z$-dependence and no $z$-component. So $\vec{G}$ is curl-free on the plane minus the origin. Encouraged, you try to build $\vec{F}$.

The simplest extension is $\vec{F}(x,y,z) = \vec{G}(x,y)$, constant in $z$. Its curl is zero wherever it is defined — but it is defined only off the whole $z$-axis, not merely off the origin. Condition (i) fails at every $(0,0,z)$ with $z \neq 0$.

So you look for a potential. You recognise $\vec{G}$ as a rescaled angular field: with $\theta = \arg(x+2iy)$, i.e. $\tan\theta = 2y/x$, differentiation gives $\partial_x(\frac{1}{2}\theta) = -y/D$ and $\partial_y(\frac{1}{2}\theta) = x/D$, so $\vec{G} = \nabla(\frac{1}{2}\theta)$ locally. You now try to extend $\frac{1}{2}\theta$ into space as some $\varphi(x,y,z)$ that is smooth away from the origin and set $\vec{F} = \nabla\varphi$. Every attempt runs into the same problem: $\theta$ is not single-valued — it increases by $2\pi$ round the ellipse — and any extension inherits the jump somewhere. You can move the bad set around, but you cannot shrink it to a point. At this stage many students either assert "yes, by extending the potential" (wrong) or stop, sensing the obstacle without being able to prove it is an obstacle.

*The model chain.* Suppose $\vec{F}$ exists. Let $E$ be the ellipse $x^2+4y^2=4$, $z=0$, parametrised by $\vec{r}(\theta) = (2\cos\theta, \sin\theta, 0)$, $0 \le \theta \le 2\pi$. Let $S$ be the upper half of the ellipsoid $x^2+4y^2+z^2=4$, $z \ge 0$, oriented by the upward normal, so that $\partial S = E$ traversed with $\theta$ increasing.

$S$ does not contain the origin: its points with $z = 0$ satisfy $x^2 + 4y^2 = 4$, and its points with $z > 0$ are not the origin. $S$ is compact, so $\vec{F}$ is continuously differentiable on an open set containing $S$ by (i). Stokes' theorem therefore applies:

$$\oint_E \vec{F} \cdot d\vec{r} = \iint_S (\operatorname{Curl} \vec{F}) \cdot \vec{n}\, dS = 0$$

by (ii). On $E$ we have $z = 0$, so by (iii) $\vec{F} = \vec{G}$ there. On $E$, $x^2 + 4y^2 = 4\cos^2\theta + 4\sin^2\theta = 4$, so

$$\vec{G}(\vec{r}(\theta)) = \left(\frac{-\sin\theta}{4}, \frac{2\cos\theta}{4}, 0\right), \qquad \vec{r}\,'(\theta) = (-2\sin\theta, \cos\theta, 0),$$

and

$$\oint_E \vec{G} \cdot d\vec{r} = \int_0^{2\pi} \left(\frac{2\sin^2\theta}{4} + \frac{2\cos^2\theta}{4}\right) d\theta = \int_0^{2\pi} \frac{1}{2}\, d\theta = \pi.$$

So $0 = \pi$, a contradiction. No such $\vec{F}$ exists.

**Layer 2 THE HOW — the bifurcation**

**Bifurcating point.** Both chains compute that $\vec{G}$ is curl-free in the punctured plane. The natural attempt reads that as permission ("locally a gradient, so extend"). The model reads it as irrelevant, and asks instead for the one number a curl-free extension would force to be zero: the circulation of $\vec{G}$ round a loop that encircles the origin in the plane.

> *Curl zero is a local condition, and $\vec{G}$ passes it. The question is whether the* shape *of the domain lets me turn "curl zero" into "circulation zero". In the plane minus a point it does not. In space minus a point it does, because any loop bounds a surface that dodges the point. So I should compute a circulation.*

> *The field looks like $d\theta$ for an ellipse. The right loop is the level curve of $x^2 + 4y^2$, the ellipse $(2\cos\theta, \sin\theta)$, because there the denominator is constant and the integral collapses.*

> *To use Stokes I need a surface with that ellipse as boundary and no origin on it. Cap the ellipse with a dome.*

**Layer 3 THE WHY**

Why is circulation the right object? Go back to what curl *is*. Stokes' theorem is the definition in integrated form: curl is circulation per unit area, and the integral of curl over a surface is the circulation round its boundary. So "Curl $\vec{F} = \vec{0}$ on a region" means "every loop that bounds a surface inside the region has zero circulation". The problem hands you a field on the plane and asks whether it can sit inside a curl-free field on space minus a point. The only thing that can go wrong is a loop with non-zero circulation that bounds a surface in the bigger space. That is precisely the thing to test.

Why does the plane allow what space forbids? In $\mathbb{R}^2 \setminus \{0\}$, a loop round the origin bounds no region avoiding the origin: any flat disc spanning it contains the origin. So Stokes (Green) says nothing, and a curl-free field is free to have circulation — the angle field $d\theta$ is the standard example, with circulation $2\pi$. In $\mathbb{R}^3 \setminus \{0\}$, the same loop bounds a dome that lifts over the origin. One point is too small to catch a surface; you can always push the surface off it. (A line, such as the $z$-axis, is not too small, which is why the naive extension $\vec{F}(x, y, z) = \vec{G}(x, y)$ needs the whole $z$-axis removed.) That single geometric contrast — "a point can be dodged in 3D but not in 2D" — is the whole problem.

Why the ellipse and not the unit circle? The field is built from $D = x^2 + 4y^2$, so its natural level sets are ellipses. On the unit circle $D = \cos^2\theta + 4\sin^2\theta$ varies and the integral is an unpleasant rational-trigonometric integral (it still equals $\pi$, since all loops round the origin in the plane have the same circulation for a curl-free field, but you would have to prove that too). On the ellipse $D \equiv 4$ and the integrand is the constant $\frac{1}{2}$. Choose the curve adapted to the denominator. That is cheapness, not luck.

Why is the answer $\pi$ and not $2\pi$? $\vec{G} = \nabla(\frac{1}{2}\theta)$ where $\theta = \arg(x + 2iy)$. Going once round, $\theta$ increases by $2\pi$, so $\frac{1}{2}\theta$ increases by $\pi$. The non-zero value is the winding of the argument — the same obstruction the natural attempt met when the potential refused to be single-valued. The natural attempt had found the right phenomenon; it lacked the tool (Stokes on a dodging surface) to turn the phenomenon into a proof.

Kedlaya–Poonen–Vakil's solution begins by choosing "a surface not containing $(0,0,0)$ whose boundary is the ellipse". Here is where that comes from: the hypothesis (ii) is only usable through Stokes, Stokes needs a surface, and the one constraint the surface must satisfy is to avoid the single point where (ii) is not given.

**Layer $+\alpha$ BACKTESTING**

**1.** The first useful cue is the mismatch of domains: $\vec{G}$ is singular at the origin of the plane, and $\vec{F}$ is required to be smooth and curl-free off the origin of space. On seeing it, write down: "Curl $= 0$ off a point in $\mathbb{R}^3 \Rightarrow$ every loop has zero circulation; compute $\oint \vec{G} \cdot d\vec{r}$ round the origin".

**2.** Stokes' theorem applied to a surface spanning a planar loop and bowing out of the plane to miss the origin; the loop chosen as a level set of the denominator.

**3.** Hesitation occurs after verifying that $\vec{G}$ is curl-free in the plane: it feels like the answer is "yes". Reinforce with: curl zero is local; whether it gives a potential depends on the shape of the domain; test a loop.

**4.** The trap is the local computation $\partial_x G_2 - \partial_y G_1 = 0$, which is correct and seductive. A second trap is the extension $\vec{F}(x,y,z) = \vec{G}(x,y)$, which is curl-free but singular on the whole $z$-axis, violating (i). A third is picking the unit circle and facing a hard integral.

**5.** Whenever a problem contrasts "a field with a singularity in the plane" against "a field on space minus a point", or more generally asks whether a locally conservative field extends to a bigger domain, compute a circulation round the singularity and ask whether the loop bounds a surface in the bigger domain.

**6.** Reusable rule: curl-free implies zero circulation on every loop that bounds a surface inside the domain. To disprove existence of a curl-free (or conservative) field, exhibit one such loop with non-zero circulation. Choose the loop as a level set of whatever makes the integrand constant.

▷ **Cue:** a planar field like $\frac{(-y,x)}{\text{quadratic form}}$ $\longrightarrow$ **Move:** it is $\nabla$ of a (scaled) angle; expect non-zero circulation round the origin

▷ **Cue:** asked whether a curl-free field extends to a larger domain $\longrightarrow$ **Move:** find a loop with non-zero circulation that bounds a surface in the larger domain; Stokes gives the contradiction

▷ **Cue:** domain is $\mathbb{R}^3$ minus a point $\longrightarrow$ **Move:** every loop bounds a surface avoiding the point (cap it with a dome); curl-free fields there are conservative

▷ **Cue:** a line integral of a field with denominator $x^2 + 4y^2$ $\longrightarrow$ **Move:** integrate round the ellipse $(2\cos\theta, \sin\theta)$ where the denominator is constant

▷ **Cue:** curl computed to be zero in a punctured domain $\longrightarrow$ **Move:** do not conclude a potential exists; check the circulation round the puncture

## 1987 A6

**1987 A6.** For each positive integer $n$, let $a(n)$ be the number of zeroes in the base 3 representation of $n$. For which positive real numbers $x$ does the series

$$\sum_{n=1}^{\infty} \frac{x^{a(n)}}{n^3}$$

converge?

**Fields:** analysis (series), combinatorics of digits **Position:** A6 — last on the paper and scored as hard (few full marks), but the difficulty is entirely in choosing how to group the terms; after that it is a geometric series **Answer:** The series converges exactly for $0 < x < 25$. Grouping the $n$ with $k+1$ base-3 digits, $\sum x^{a(n)} = 2(x+2)^k$ over the block and $n^3 \approx 27^k$, so the series behaves like $\sum\left(\frac{x+2}{27}\right)^k$.

## Layer 1 THE WHAT — surface analysis

*The natural attempt.* You start with comparison tests on individual terms. For $x \le 1$ you have $x^{a(n)} \le 1$, so the series is dominated by $\sum 1/n^3$ and converges. For $x > 1$ the numerator can grow, and you look for its worst case: $a(n) \le \log_3 n$, so $x^{a(n)} \le x^{\log_3 n} = n^{\log_3 x}$, and the term is at most $n^{\log_3 x - 3}$. That converges when $\log_3 x - 3 < -1$, i.e. $x < 9$.

So you have convergence for $x < 9$. Now you want divergence above some threshold, and you look at the numbers with the most zeros: $n = 3^k$ has $a(n) = k$, giving terms $x^k/27^k$. Along that subsequence the terms fail to tend to 0 once $x \ge 27$. So you have convergence below 9, divergence at or above 27, and a gap between. Many students guess one of the endpoints. Both guesses are wrong.

The wall is structural. Termwise bounds use only the extreme values of $a(n)$, one number at a time. But whether the series converges depends on how *many* $n$ have *how many* zeros — a counting question that single-term comparisons cannot see.

*The model chain.* All terms are positive, so the series converges if and only if any grouping of consecutive terms gives a convergent series of block sums.

*Blocks.* A positive integer $n$ has exactly $k+1$ base-3 digits if and only if $3^k \le n < 3^{k+1}$. Define, for $k \ge 0$,

$$S_k = \sum_{n=3^k}^{3^{k+1}-1} \frac{x^{a(n)}}{n^3}, \qquad T_k = \sum_{n=3^k}^{3^{k+1}-1} x^{a(n)}.$$

In the block $27^k \le n^3 < 27^{k+1}$, so

$$\frac{T_k}{27^{k+1}} \le S_k \le \frac{T_k}{27^k}.$$

*Counting.* A $(k+1)$-digit base-3 number has a leading digit in $\{1,2\}$ and $k$ further digits in $\{0,1,2\}$. Each further digit contributes a factor $x$ to $x^{a(n)}$ if it is 0 and a factor 1 if it is 1 or 2. Summing over all choices, the digits contribute independently:

$$T_k = 2 \cdot (x+1+1)^k = 2(x+2)^k.$$

(Equivalently, the number of such $n$ with exactly $i$ zeros is $\binom{k}{i}2^{k+1-i}$, and $\sum_i \binom{k}{i}2^{k+1-i}x^i = 2(x+2)^k$.)

*Conclusion.* Hence

$$\frac{2}{27}\left(\frac{x+2}{27}\right)^k \le S_k \le 2\left(\frac{x+2}{27}\right)^k.$$

If $x < 25$ then $r = (x+2)/27 < 1$ and $\sum S_k \le 2\sum r^k < \infty$. If $x \ge 25$ then $r \ge 1$ and $S_k \ge \frac{2}{27}$ for every $k$, so $\sum S_k$ diverges. The series converges exactly for $0 < x < 25$.

**Caveat.** Kedlaya–Poonen–Vakil write $T_k = (x+2)^k$; the binomial sum they display evaluates to $2(x+2)^k$ (the leading digit has two choices). I checked the count numerically for $k \leq 4$. The missing factor 2 does not affect convergence or the answer $x < 25$.

### Layer 2 THE HOW — the bifurcation

**Bifurcating point.** Both chains agree on the easy part ($x \leq 1$ converges). They split at the choice of how to control $x^{a(n)}$. The natural attempt bounds each term separately by its worst case. The model groups all $n$ of the same length, where $n^3$ is essentially constant, and sums the numerators exactly.

> *The numerator depends on the digits and the denominator depends on the size. Inside a block of numbers with the same number of digits, the size hardly changes — $n^3$ moves by a factor of 27 at most. So I should freeze the denominator on each block and count the numerators exactly.*

> *Summing $x^{\text{number of zeros}}$ over all digit strings is a product: each free digit contributes $x+1+1$. It is the binomial theorem in disguise.*

### Layer 3 THE WHY

Why group by number of digits? The quantity $a(n)$ is defined through the base-3 representation, and that representation has a natural unit: the length. Numbers of the same length are exactly the interval $[3^k, 3^{k+1})$, on which $n$ varies by a bounded factor. So the one piece of the term that is hard to track (the numerator) is organised by length, and the piece that is easy to track (the denominator) is almost constant on each length class. Grouping by length puts both pieces into the same coordinates. This is the Cauchy condensation idea: when a sequence has a natural scale $3^k$, test convergence one scale at a time.

Why is the block sum a product? Because the digits are independent choices. Summing a function of the form $\prod_j g(d_j)$ over all strings $d_1 \dots d_k$ gives $\prod_j \left(\sum_d g(d)\right)$ — the distributive law, nothing more. Here $g(0) = x$ and $g(1) = g(2) = 1$, so each free digit contributes $x+2$. Once you see $x^{a(n)}$ as a product over digits ($x$ for each zero), the count is immediate.

Why did the termwise bounds give the wrong window? They compared the *largest* numerator in a block, $x^k$ (attained once, at $3^k$), and the *typical* numerator, 1, against $27^k$. The truth is the *total*, $2(x+2)^k$, which lies between them: there are many numbers with a moderate number of zeros, and their combined weight is what decides convergence. The threshold $x+2 = 27$ says that the average numerator must grow more slowly than $27^k/3^k \cdot 3^k$ — equivalently, the block sum of numerators must grow more slowly than the size of the denominator.

The same argument in base $b$ with $n^b$ in the denominator gives $T_k = (b-1)(x+b-1)^k$ against $b^{bk}$, hence convergence iff $x < b^b - b + 1$; for $b = 3$ that is 25. Kedlaya–Poonen–Vakil state this generalisation as a remark.

### Layer $+\alpha$ BACKTESTING

**1.** The first useful cue is "the number of zeroes in the base 3 representation". On seeing it, write down the blocks $3^k \leq n < 3^{k+1}$ and the sum $T_k = \sum x^{a(n)}$ over one block.
**2.** Group terms by number of digits (condensation), bound $n^3$ between $27^k$ and $27^{k+1}$ on each block, and evaluate $T_k$ as a product over digits.
**3.** Hesitation occurs after termwise comparison leaves the gap $9 \leq x < 27$. Reinforce with: a digit statistic needs a count over all digit strings, not a bound on one of them.
**4.** The trap is termwise comparison, which gives correct but non-matching bounds and invites a guess at 9 or 27. A second trap is forgetting that the leading digit cannot be 0.
**5.** Whenever a series weights $n$ by a function of its digits, group by length in that base and sum over digit strings by the distributive law.

**6.** Reusable rule: when the terms have a natural scale (lengths, powers of a base, dyadic blocks), test convergence one scale at a time, freezing the slowly varying part of the term and summing the rest exactly.

▷ **Cue:** a series whose terms depend on the digits of $n$ in base $b$ $\longrightarrow$ **Move:** group $n$ into blocks $b^k \le n < b^{k+1}$ and treat the denominator as constant on each block

▷ **Cue:** sum of $x^{(\text{number of digits equal to } d)}$ over all digit strings of length $k$ $\longrightarrow$ **Move:** product over digits: $(x + b - 1)^k$, by the distributive law

▷ **Cue:** termwise comparison bounds leave a gap between convergence and divergence $\longrightarrow$ **Move:** the answer depends on a count, not an extreme value; sum over the natural blocks

▷ **Cue:** a positive series with a natural scale $b^k$ $\longrightarrow$ **Move:** condensation: convergence is equivalent to convergence of the block sums

▷ **Cue:** divergence to show at the boundary value $\longrightarrow$ **Move:** show block sums do not tend to 0

# Putnam 1987 — Paper B

## 1987 B1

**1987 B1.** Evaluate
$$\int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}.$$

**Fields:** integration, symmetry of definite integrals **Position:** B1 – opening problem of the second session **Answer:** The integral equals 1.

### Layer 1 THE WHAT — surface analysis

*The natural attempt.* You see a definite integral and reach for an antiderivative. The integrand is a quotient of square roots of logarithms, so you look for a substitution that tames it. Try $u = \ln(9-x)$: then $x = 9-e^u$, $dx = -e^u\,du$, and $\ln(x+3) = \ln(12-e^u)$, so the integral becomes
$$\int_{\ln 5}^{\ln 7} \frac{\sqrt{u}\,e^u}{\sqrt{u}+\sqrt{\ln(12-e^u)}}\,du,$$
which is worse: an exponential has appeared and the second logarithm has not gone away. Try instead to rationalise the denominator by multiplying through by $\sqrt{\ln(9-x)}-\sqrt{\ln(x+3)}$: the integrand becomes
$$\frac{\ln(9-x)-\sqrt{\ln(9-x)\ln(x+3)}}{\ln(9-x)-\ln(x+3)},$$
and now the denominator vanishes at $x=3$ (where $9-x=x+3=6$), so you have manufactured a removable singularity in the middle of the interval and a square root of a product of logarithms on top. Integration by parts has nothing to grip. After ten minutes you have three rewritings, none of which is closer to an antiderivative, and the suspicion that $\sqrt{\ln}$ has no elementary antiderivative in any combination of this shape. The suspicion is correct: this route does not end.

*The model chain.* Write $g(x)=\sqrt{\ln(9-x)}$ and $h(x)=\sqrt{\ln(x+3)}$. For $2\le x\le 4$ we have $9-x\in[5,7]$ and $x+3\in[5,7]$, so both logarithms are at least $\ln 5>0$; hence $g,h$ are continuous and positive on $[2,4]$ and the integrand $f=g/(g+h)$ is continuous there. Let $I=\int_2^4 f(x)\,dx$.

The two arguments $9-x$ and $x+3$ add to 12, and as $x$ runs from 2 to 4 one runs from 7 down to 5 while the other runs from 5 up to 7. The reflection $x\mapsto 6-x$ fixes the interval $[2,4]$ (it swaps the endpoints and fixes the midpoint 3) and swaps the arguments:
$$9-(6-x)=x+3, \qquad (6-x)+3=9-x.$$
So $g(6-x)=h(x)$ and $h(6-x)=g(x)$. Substituting $x=6-y$, $dx=-dy$, and reversing the limits,
$$I=\int_2^4 \frac{g(6-y)}{g(6-y)+h(6-y)}\,dy=\int_2^4 \frac{h(y)}{h(y)+g(y)}\,dy.$$

Adding this to the original expression for $I$ (renaming the dummy variable),

$$2I = \int_2^4 \frac{g(x)+h(x)}{g(x)+h(x)}\,dx = \int_2^4 1\,dx = 2,$$

so $I = 1$. (Numerical quadrature gives 1.0000 to machine precision.)

**Layer 2 THE HOW — the bifurcation**

**Bifurcating point.** Both paths start by looking at the integrand. The natural attempt asks "what is an antiderivative of this?" and starts transforming the integrand in isolation. The model asks "what does this integrand look like at the other end of the interval?" and compares $f(x)$ with $f(6-x)$. The split is at the very first move: the natural attempt treats the limits 2 and 4 as numbers to plug in at the end; the model treats them as part of the problem's structure from the start.

> *Nine minus x and x plus three – one goes down while the other goes up, and they cross at $x = 3$, the midpoint of $[2,4]$. If I reflect the interval about 3, the two logarithms swap places. Then the integrand turns into its complement, and I can add the two versions instead of integrating either one.*

> *This integrand has no antiderivative I can write down, and the examiners know that. A definite integral with a hopeless integrand and conveniently chosen limits is asking me to use the limits, not to integrate.*

**Layer 3 THE WHY**

Why is the reflection natural and not a trick? Start from what a definite integral is: a number depending on the integrand *and* the interval. Every symmetry of the interval is a free substitution – $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ for any integrable $f$, because $x \mapsto a+b-x$ is a bijection of $[a,b]$ with $|dx|$ preserved. This identity costs nothing and holds for every integral you will ever meet. The only question is whether $f(a+b-x)$ is related to $f(x)$ in a useful way. So the reflection is not something one "notices"; it is the first thing to test whenever the antiderivative is out of reach, because it is always available.

What tells you it will work here, before you try it? The arguments of the two logarithms. Their sum $(9-x)+(x+3) = 12$ is constant, so they are reflections of each other about 6; and the interval $[2,4]$ is exactly where each runs through $[5,7]$ in opposite directions. A problem-setter who wanted an honest integral would not choose limits that make the two arguments trade places. The phrasing of the problem – two expressions in $x$ with opposite-sign $x$-coefficients, limits centred where they are equal – is the cue.

Now the structural fact that makes the reflection finish the problem. The integrand has the form $\dfrac{A}{A+B}$. Any expression of that form has a complement $\dfrac{B}{A+B}$ with which it sums to 1. If a symmetry swaps $A$ and $B$, it sends the integrand to its complement, and averaging over the symmetry turns the integrand into the constant $\frac{1}{2}$. That is the whole mechanism: the integral of $f$ equals the integral of its average with its reflection, and that average is $\frac{1}{2}$, so $I = \frac{1}{2}(4-2) = 1$. The square roots and logarithms are decoration; the book's remark says the same argument works for any continuous $\varphi$ in place of $\sqrt{\ln}$, provided $\varphi(9-x)+\varphi(x+3) \neq 0$ on the interval. Once you see that the answer does not depend on which function sits inside, you understand why the antiderivative route was doomed: the problem was never about $\sqrt{\ln}$.

Kedlaya's solution says "this symmetry suggests the substitution $x = 6-y$". The 6 comes from $a+b = 2+4$; equivalently, from solving $9-x = x+3$ to find the fixed point $x = 3$ and reflecting about it. Either derivation gives the same substitution, and the second one shows why it swaps the arguments.

The only rigour point is continuity: $\ln(9-x)$ and $\ln(x+3)$ are positive on $[2,4]$, so the denominator never vanishes and each integral is a proper Riemann integral. State it; it is one line.

**Layer $+\alpha$ BACKTESTING**

**1.** The first cue is the pair $9-x$ and $x+3$ inside matching functions, together with limits $2,4$ whose midpoint 3 is where $9-x=x+3$. On seeing it, write down $x \mapsto 2+4-x=6-x$ and compute what it does to each argument.

**2.** It suggests the reflection substitution $x \mapsto a+b-x$, followed by adding the reflected integral to the original (the "King's property" of definite integrals).

**3.** Hesitation occurs when the first substitution attempt fails and the student keeps looking for a cleverer substitution for an antiderivative. What reinforces it away is the rule: if the integrand has no visible antiderivative and the limits are specific numbers, test the reflection before anything else.

**4.** The trap is the antiderivative hunt: $\sqrt{\ln u}$ has no elementary antiderivative, and rationalising the denominator introduces a removable singularity at $x=3$ that makes things look worse. Zombie maths is punished completely – there is no brute-force route.

**5.** When the integrand has the shape $\dfrac{\varphi(u(x))}{\varphi(u(x))+\varphi(v(x))}$ and $u, v$ trade values across the interval, the answer is half the length of the interval. More generally: a definite integral whose integrand you cannot integrate is a question about the interval's symmetry.

**6.** The reusable decision rule: before hunting for an antiderivative, compute $f(a+b-x)$. If $f(x)+f(a+b-x)$ is simpler than $f$, the integral is $\frac{1}{2}\int_a^b \big(f(x)+f(a+b-x)\big)dx$.

▷ **Cue:** two expressions in $x$ with opposite-sign $x$-coefficients, e.g. $9-x$ and $x+3$, whose sum is constant, inside the same function $\longrightarrow$ **Move:** find where they are equal; reflect the variable about that point; check that the reflection swaps them

▷ **Cue:** an integrand of the form $A/(A+B)$ where a substitution swaps $A$ and $B$ $\longrightarrow$ **Move:** add the original integral to the substituted one; the integrand becomes 1 and the answer is half the interval length

▷ **Cue:** a definite integral with no elementary antiderivative but specific, chosen-looking limits $\longrightarrow$ **Move:** apply $x \mapsto a+b-x$ before attempting any antiderivative

## 1987 B2

**1987 B2.** Let $r, s$ and $t$ be integers with $0 \le r$, $0 \le s$ and $r+s \le t$. Prove that

$$\frac{\binom{s}{0}}{\binom{t}{r}} + \frac{\binom{s}{1}}{\binom{t}{r+1}} + \cdots + \frac{\binom{s}{s}}{\binom{t}{r+s}} = \frac{t+1}{(t+1-s)\binom{t-s}{r}}.$$

**Fields:** combinatorics, binomial identities, induction **Position:** B2 – second problem of the second session **Answer:** The identity holds; proof by induction on $s$ via Pascal's rule on the numerators, or by converting each quotient into a product of binomials and counting.

**Layer 1 THE WHAT — surface analysis**

Throughout, write

$$F(r,s)=\sum_{i=0}^{s}\frac{\binom{s}{i}}{\binom{t}{r+i}}, \qquad G(r,s)=\frac{t+1}{(t+1-s)\binom{t-s}{r}},$$

with $t$ fixed, defined whenever $r,s \ge 0$ and $r+s \le t$. Note $t+1-s \ge r+1 \ge 1$ and $\binom{t-s}{r} \ge 1$, so $G$ is well defined.

*The natural attempt.* You check small cases. For $s = 0$ both sides are $1/\binom{t}{r}$. For $s = 1$ the left side is $\dfrac{1}{\binom{t}{r}} + \dfrac{1}{\binom{t}{r+1}}$; writing both over factorials,

$$\frac{r!\,(t-r)!}{t!} + \frac{(r+1)!\,(t-r-1)!}{t!} = \frac{r!\,(t-r-1)!}{t!}\big[(t-r)+(r+1)\big] = \frac{(t+1)\,r!\,(t-r-1)!}{t!},$$

and the right side is $\dfrac{t+1}{t\binom{t-1}{r}} = \dfrac{(t+1)\,r!\,(t-1-r)!}{t\,(t-1)!}$ – the same. Encouraged, you try the general case the same way: put every term over the common factor $\frac{r!(t-r-s)!}{t!}$ and look at what is left. The $i$-th term leaves $\binom{s}{i}\frac{(r+i)!}{r!}\frac{(t-r-i)!}{(t-r-s)!}$, a product of two rising factorials times a binomial, and you are looking at a sum of $s+1$ such products with no evident way to add them. You try to telescope: is there an $H(i)$ with $H(i+1) - H(i)$ equal to the $i$-th term? Nothing presents itself. You try induction on $s$ by peeling off the last term, $F(r,s) = ? + \binom{s}{s}/\binom{t}{r+s}$, and stall because the numerators $\binom{s}{i}$ all change when $s$ changes, so $F(r,s)$ minus its last term is not $F(r,s-1)$. The wall is real: the reciprocal binomials in the denominators move with $i$, and a sum of fractions with moving denominators does not simplify by grinding.

*The model chain. Induction on $s$, via Pascal's rule on the numerators.* Base case $s = 0$: $F(r,0) = 1/\binom{t}{r}$ and $G(r,0) = \frac{t+1}{(t+1)\binom{t}{r}}$, equal.

Let $s \geq 1$ and assume $F(r', s-1) = G(r', s-1)$ for all $r' \geq 0$ with $r' + s - 1 \leq t$. Apply $\binom{s}{i} = \binom{s-1}{i} + \binom{s-1}{i-1}$ (with $\binom{s-1}{s} = \binom{s-1}{-1} = 0$):

$$F(r,s) = \sum_{i=0}^{s-1} \frac{\binom{s-1}{i}}{\binom{t}{r+i}} + \sum_{i=1}^{s} \frac{\binom{s-1}{i-1}}{\binom{t}{r+i}} = F(r,s-1) + F(r+1,s-1),$$

the second sum after re-indexing $i \mapsto i+1$. Both are covered by the hypothesis, since $r + (s-1) \leq t$ and $(r+1) + (s-1) \leq t$. So it remains to check that $G$ satisfies the same recursion. Put $m = t + 1 - s$ ($\geq r+1$). From factorials,

$$\binom{m}{r} = \frac{m}{m-r}\binom{m-1}{r}, \qquad \binom{m}{r+1} = \frac{m}{r+1}\binom{m-1}{r},$$

and $m - 1 = t - s$, so

$$G(r,s-1) + G(r+1,s-1) = \frac{t+1}{(m+1)}\left[\frac{1}{\binom{m}{r}} + \frac{1}{\binom{m}{r+1}}\right] = \frac{t+1}{(m+1)\,m\binom{t-s}{r}}\big[(m-r)+(r+1)\big] = \frac{t+1}{m\binom{t-s}{r}} = G(r,s)$$

Hence $F(r,s) = G(r,s)$, completing the induction.

*Why the identity is true (second proof).* Rewrite each quotient as a product. Since $(t-r-i) - (t-r-s) = s - i$,

$$\frac{\binom{s}{i}}{\binom{t}{r+i}} = \frac{s!\,(r+i)!\,(t-r-i)!}{i!\,(s-i)!\,t!} = \frac{s!\,r!\,(t-r-s)!}{t!}\binom{r+i}{r}\binom{t-r-i}{t-r-s}.$$

So the identity is equivalent to

$$\sum_{j=r}^{r+s} \binom{j}{r}\binom{t-j}{t-r-s} = \binom{t+1}{s} \qquad (j = r+i), \tag{$*$}$$

because then $F = \frac{s!\,r!\,(t-r-s)!}{t!} \cdot \frac{(t+1)!}{s!\,(t+1-s)!} = \frac{(t+1)\,r!\,(t-s-r)!}{(t+1-s)\,(t-s)!} = G$. Both sides of $(*)$ count strings of $t+1$ symbols consisting of $s$ zeros, $t-s$ ones and one comma, with exactly $r$ ones before the

comma. Right side: choose the $s$ positions of the zeros among $t+1$; the remaining $t+1-s$ positions must read (ones, comma, ones) with the comma in position $r+1$ of them, which is possible since $r \le t-s$. Left side: let $j$ be the number of digits before the comma. Exactly $r$ of them are ones, so $j-r$ of them are zeros and $r \le j \le r+s$; there are $\binom{j}{r}$ ways to fill them and $\binom{t-j}{t-s-r}$ ways to place the remaining $t-s-r$ ones among the $t-j$ digits after the comma. (Exhaustive check: both the identity and $(*)$ hold exactly for all $0 \le t \le 13$.)

**Layer 2 THE HOW — the bifurcation**

**Bifurcating point.** Both paths agree that small cases hold and that induction on $s$ is the natural frame. They split at the step "relate $F(r,s)$ to smaller cases". The natural attempt peels a *term* off the sum (the last one), which fails because every numerator depends on $s$. The model splits every *numerator* by Pascal's rule, which turns one sum into two sums of the same shape – one with $r$ unchanged, one with $r$ shifted – and so demands a two-variable induction.

> *Every numerator is $\binom{s}{i}$, and $s$ is the thing I am inducting on. I cannot remove a term, but I can split every term: $\binom{s}{i} = \binom{s-1}{i} + \binom{s-1}{i-1}$. The shifted half has its denominators shifted too, which is the same sum with $r+1$ in place of $r$. So I induct on $s$ for all $r$ at once.*

> *A sum of reciprocals of binomials is ugly; a sum of products of binomials is a convolution, and convolutions have closed forms. Let me clear the factorials and see whether the quotient is secretly a product.*

**Layer 3 THE WHY**

Why is Pascal on the numerator natural rather than lucky? Look at what an induction on $s$ needs: an expression for $F(r,s)$ in terms of $F(\cdot, s-1)$. The only place $s$ enters the left side is the numerators $\binom{s}{i}$ and the length of the sum. Pascal's rule is *the* recursion that lowers the top index of a binomial coefficient – it is the definition of the triangle. So "induct on $s$" plus "$s$ sits on top of a binomial" forces Pascal. Once you apply it, the shift $i \mapsto i-1$ in one half moves the denominator from $\binom{t}{r+i}$ to $\binom{t}{r+1+i}$, which is why $r$ must be free in the hypothesis. This is the standard strengthening: when the recursion changes a second parameter, the inductive hypothesis must quantify over that parameter. Kedlaya's Solution 1 writes $F(r,s,t) = F(r,s-1,t) + F(r+1,s-1,t)$ without comment; this is where it comes from.

The right side then has to satisfy the same Pascal-type recursion, and the verification is an algebraic check, not an insight. The cheap way to do it is to express both $\binom{m}{r}$ and $\binom{m}{r+1}$ as multiples of the single coefficient $\binom{m-1}{r}$, so the bracket collapses to $(m-r)+(r+1) = m+1$ and cancels the $m+1$ outside. That cancellation is the sign that the identity is right; if it had not happened, you would recheck the statement before the algebra.

The second proof answers the deeper question: why should a sum of reciprocals of binomials have a closed form at all? The denominator $\binom{t}{r+i}$ and the numerator $\binom{s}{i}$ together are, up to a constant, $\binom{r+i}{r}\binom{t-r-i}{t-r-s}$ – a quotient of binomial coefficients with the same factorials rearranged. The structural fact is that $\frac{\binom{s}{i}}{\binom{t}{r+i}}$ and $\frac{\binom{r+i}{r}\binom{t-r-i}{s-i}}{\binom{t}{s}\binom{t-s}{r}}$ are two ways of writing the same ratio of multinomials. Products of binomials summed over a split point $j$ are the signature of "count objects by where a marker sits"; the comma string is the marker made literal. That is also why the generating-function proof in the book works: $\sum_j \binom{j}{r} x^{j-r} = (1-x)^{-(r+1)}$, and multiplying two such series is exactly the convolution in $(*)$.

Which proof to write in the exam? The induction is cheaper to discover under pressure and needs no combinatorial model; the counting proof explains the identity and would be the choice if the induction's algebra refused to close.

**Layer $+\alpha$ BACKTESTING**

**1.** The first useful cue is that $s$ appears only as the top of $\binom{s}{i}$ and as the length of the sum. On seeing it, write Pascal's rule $\binom{s}{i} = \binom{s-1}{i} + \binom{s-1}{i-1}$ under the sum.
**2.** It suggests induction on $s$, with the hypothesis holding for all admissible $r$ simultaneously, followed by checking that the closed form obeys the same recursion.
**3.** Hesitation occurs after splitting, on seeing $r+1$ appear in the second sum: it looks like the recursion has left the family. It is reinforced away by strengthening the hypothesis to all $r$ and checking the constraint $(r+1)+(s-1) \le t$ still holds.
**4.** The trap is to peel off the last term of the sum, as for a partial sum, or to grind everything into factorials and attempt a direct summation. Both founder on the moving denominators.
**5.** A reciprocal of a binomial coefficient multiplied by another binomial is often a product of binomials in disguise; rewrite into factorials and regroup before summing. Any recursion that shifts a second parameter means induct over the first for all values of the second.
**6.** The reusable decision rule: to induct on a parameter sitting on top of a binomial, split with Pascal's rule and let the induction carry every other parameter; then verify the target satisfies the same recursion.

▷ **Cue:** the induction variable $s$ appears only as the top index $\binom{s}{i}$ inside a sum $\longrightarrow$ **Move:** split every term with Pascal's rule, re-index, and recognise two copies of the same sum with shifted parameters
▷ **Cue:** a recursion $F(r,s) = F(r,s-1) + F(r+1,s-1)$ that changes $r$ as well as $s$ $\longrightarrow$ **Move:** induct on $s$ with the hypothesis quantified over all admissible $r$
▷ **Cue:** a sum of quotients $\binom{s}{i}/\binom{t}{r+i}$ with moving denominators $\longrightarrow$ **Move:** rewrite in factorials; regroup into $\binom{r+i}{r}\binom{t-r-i}{s-i}$-type products; recognise a convolution
▷ **Cue:** $\sum_j \binom{j}{a}\binom{n-j}{b}$ $\longrightarrow$ **Move:** it equals $\binom{n+1}{a+b+1}$: count binary strings with a marker, split by the marker's position

## 1987 B3

**1987 B3.** Let $F$ be a field in which $1+1 \neq 0$. Show that the set of solutions to the equation $x^2+y^2=1$ with $x$ and $y$ in $F$ is given by $(x,y)=(1,0)$ and

$$(x,y) = \left(\frac{r^2-1}{r^2+1}, \frac{2r}{r^2+1}\right)$$

where $r$ runs through the elements of $F$ such that $r^2 \neq -1$.

**Fields:** algebra (fields), rational parametrisation of conics **Position:** B3 – third problem of the second session **Answer:** Every solution other than $(1,0)$ arises from $r = y/(1-x)$, which satisfies $r^2 \neq -1$ and reproduces $(x,y)$; conversely each listed point is a solution.

**Layer 1 THE WHAT — surface analysis**

For $r \in F$ with $r^2 \neq -1$ write $(x_r, y_r) = \left(\frac{r^2-1}{r^2+1}, \frac{2r}{r^2+1}\right)$. Two inclusions are needed: every listed point is a solution, and every solution is listed.

*The natural attempt.* The first inclusion is a computation, and you do it: $(r^2-1)^2 + (2r)^2 = r^4 + 2r^2 + 1 = (r^2+1)^2$, so $x_r^2 + y_r^2 = 1$; and $1^2 + 0^2 = 1$. For the converse you take a solution $(x,y)$ and try to find $r$. The formula $x = \frac{r^2-1}{r^2+1}$ looks like the thing to invert: cross-multiplying, $x(r^2+1) = r^2 - 1$, so $r^2(1-x) = 1+x$ and, for $x \neq 1$,

$$r^2 = \frac{1+x}{1-x}.$$

Now you need a square root of $\frac{1+x}{1-x}$ in $F$. Over $\mathbb{R}$ you would argue that $|x| \leq 1$ makes this non-negative and take $r = \pm\sqrt{\cdot}$, then fix the sign by looking at $y$. But $F$ is an arbitrary field: there is no ordering, "$|x| \leq 1$" means nothing, and a typical element of $F$ has no square root in $F$. You try to prove that $\frac{1+x}{1-x}$ is a square, get $\frac{1+x}{1-x} = \frac{(1+x)^2}{(1-x)(1+x)} = \frac{(1+x)^2}{1-x^2} = \frac{(1+x)^2}{y^2}$ – which *is* a square, if $y \neq 0$ – and now you have a case split on $y = 0$, a choice of sign, and a separate check that the sign agrees with $y$. It can be pushed through, but it is a proof by case analysis built around a square root that the field never promised you.

*The model chain.* *Listed points are solutions.* $1^2 + 0^2 = 1$. For $r^2 \neq -1$, $r^2 + 1 \neq 0$, so $(x_r, y_r) \in F^2$ is defined, and

$$x_r^2 + y_r^2 = \frac{(r^2-1)^2 + 4r^2}{(r^2+1)^2} = \frac{(r^2+1)^2}{(r^2+1)^2} = 1.$$

*Every solution is listed.* Let $x, y \in F$ with $x^2 + y^2 = 1$. If $x = 1$ then $y^2 = 0$, so $y = 0$ (a field has no zero divisors), and $(x, y) = (1, 0)$. Otherwise $1 - x \neq 0$; define

$$r = \frac{y}{1-x} \in F.$$

Then $y = r(1-x)$, and substituting into $y^2 = 1 - x^2 = (1-x)(1+x)$ gives $r^2(1-x)^2 = (1-x)(1+x)$. Cancelling the non-zero factor $1 - x$,

$$r^2(1-x) = 1 + x, \qquad \text{i.e.} \qquad (r^2+1)\,x = r^2 - 1. \tag{†}$$

If $r^2 = -1$, then (†) reads $0 = -2$, i.e. $1 + 1 = 0$, contrary to hypothesis. So $r^2 \neq -1$, and (†) gives $x = \frac{r^2-1}{r^2+1} = x_r$. Finally

$$y = r(1-x) = r \cdot \frac{(r^2+1) - (r^2-1)}{r^2+1} = \frac{2r}{r^2+1} = y_r.$$

So $(x, y) = (x_r, y_r)$ with $r^2 \neq -1$. (Exhaustive check over $\mathbb{F}_p$ for $p = 3, 5, 7, 11, 13, 17$: the solution set equals $\{(1,0)\} \cup \{(x_r, y_r)\}$ in every case.)

**Layer 2 THE HOW — the bifurcation**

**Bifurcating point.** Both paths do the forward verification and both, for the converse, want to express $r$ in terms of $(x, y)$. They split at the choice of which coordinate to invert. The natural attempt inverts $x = \frac{r^2-1}{r^2+1}$, which is quadratic in $r$ and hands back $r^2$, forcing a square root. The model inverts the *ratio* $\frac{y_r}{1-x_r}$, which is linear in $r$: $1 - x_r = \frac{2}{r^2+1}$ and $y_r = \frac{2r}{r^2+1}$, so $\frac{y_r}{1-x_r} = r$ exactly. That gives $r$ as a rational function of $(x, y)$, with no root to extract.

> *I am in a field with no square roots promised, so I must not solve for $r^2$. Is there a rational expression in $x$ and $y$ that returns $r$ itself? $1 - x_r$ is $2/(r^2+1)$ and $y_r$ is $2r/(r^2+1)$ – the same denominator – so their quotient is $r$. I define $r = y/(1-x)$ and check it works.*

> *Where does $1 + 1 \neq 0$ get used? It has to be used somewhere or the statement would be true without it. It is the only thing that can rule out $r^2 = -1$.*

**Layer 3 THE WHY**

Why is $r = y/(1-x)$ the natural definition and not a rabbit from a hat? There are two routes to it, and both reach "of course".

*Algebraic route.* In the statement, the map $r \mapsto (x_r, y_r)$ is given. To show it is onto (the solutions other than $(1,0)$), you need an inverse, and over an arbitrary field the only operations you have are the four field operations. So the inverse must be a rational function of $(x, y)$. The

formulas share the denominator $r^2+1$; to kill it, form a quotient of two expressions with that denominator. The numerators are $r^2-1$, $2r$, and $r^2+1$ (from 1); the combination $1-x_r = \frac{2}{r^2+1}$ has constant numerator, so $y_r/(1-x_r)$ is $r$. That is what Kedlaya's parenthetical "the reason for this is that $y_r/(1-x_r)=r$" is compressing: the search is for a ratio in which $r$ appears to the first power.

*Geometric route.* The circle contains the point $(1,0)$ with rational coordinates. A line through a point of a conic meets the conic in exactly one further point (the equation for the intersection is quadratic and one root is already known, so the other root is rational in the slope). Therefore rational points on the circle correspond to lines through $(1,0)$ with slope in $F$. The line through $(1,0)$ and $(x,y)$ has slope $\frac{y}{x-1} = -r$. This is the chord method, and the problem's parametrisation is that method written out with $r$ as the negated slope. The geometric picture is the WHY: the circle is being swept by a pencil of lines through a known point, and $r$ labels the line.

The same picture explains every feature of the statement. The point $(1,0)$ is listed separately because it is the base point of the pencil (the vertical line $x=1$ is tangent there and meets the circle only at $(1,0)$, so no other point has $x=1$). The condition $r^2 \neq -1$ excludes the lines whose "second intersection" has gone to infinity: from (†), $(r^2+1)x = r^2-1$ has no solution when $r^2+1=0$ and $2\neq 0$. And $1+1\neq 0$ is used exactly there, because in characteristic 2 the equation $x^2+y^2=1$ becomes $(x+y+1)^2=0$, a doubled line, not a circle, and the parametrisation collapses (every listed point would have $y_r=0$). A proof that never invokes $1+1\neq 0$ is wrong, and that is a check you can run on your own write-up.

The cancellation of $1-x$ in (†) is the algebraic shadow of "one root is already known": it removes the base point from the intersection.

**Layer $+\alpha$ BACKTESTING**

**1.** The first useful cue is "$F$ is a field" – no order, no square roots, only the four operations. On seeing it, write down: the inverse map must be a rational function of $(x,y)$.

**2.** It suggests looking for a ratio of the given coordinates in which the parameter appears linearly, or equivalently the chord method through the known point $(1,0)$.

**3.** Hesitation occurs at $r^2 = \frac{1+x}{1-x}$, where the student wants to take a square root. It is reinforced away by the rule that in a general field a square root is never a legal step unless the square is exhibited.

**4.** The trap is inverting the $x$-coordinate alone, which only determines $r^2$. A second trap is forgetting to use $1+1\neq 0$; it is needed to exclude $r^2=-1$, and omitting it leaves the proof incomplete.

**5.** Any rational curve given with a parametrisation having a common denominator: kill the denominator by a ratio of coordinates, and look for the ratio that is linear in the parameter. Any conic with one known point: parametrise by the slope of lines through that point.

**6.** The reusable decision rule: to prove a parametrisation is onto over a general field, find the inverse as a rational function (by a ratio, or by the chord through a known point), and then locate where each hypothesis of the problem is used.

▷ **Cue:** an arbitrary field $F$ and a claim that a parametrisation covers all solutions $\longrightarrow$ **Move:** construct the inverse using only field operations; never take a square root

▷ **Cue:** coordinates $\frac{A(r)}{D(r)}, \frac{B(r)}{D(r)}$ sharing a denominator $\longrightarrow$ **Move:** look for a ratio of affine combinations in which $D$ cancels and $r$ appears to the first power

▷ **Cue:** a conic with a known point on it $\longrightarrow$ **Move:** intersect with lines of slope $m$ through that point; the second intersection is rational in $m$

▷ **Cue:** a hypothesis such as $1+1\neq 0$ that the argument has not yet used $\longrightarrow$ **Move:** find the degenerate case it rules out; that is where it belongs in the proof

## 1987 B4

**1987 B4.** Let $(x_1, y_1) = (0.8, 0.6)$ and let $x_{n+1} = x_n \cos y_n - y_n \sin y_n$ and $y_{n+1} = x_n \sin y_n + y_n \cos y_n$ for $n = 1, 2, 3, \ldots$. For each of $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$, prove that the limit exists and find it or prove that the limit does not exist.

**Fields:** analysis; dynamical systems; trigonometry **Position:** B4 — middle of the paper; hard to start only if you miss the rotation **Answer:** Both limits exist: $x_n \to -1$ and $y_n \to 0$. The point stays on the unit circle at angle $\theta_n$, with $\theta_{n+1} = \theta_n + \sin\theta_n$, and $\theta_n$ increases to $\pi$.

### Layer 1 THE WHAT — surface analysis

*The natural attempt.* A strong student computes. $x_1 = 0.8$, $y_1 = 0.6$, so $\cos 0.6 \approx 0.8253$, $\sin 0.6 \approx 0.5646$, giving $x_2 \approx 0.8 \cdot 0.8253 - 0.6 \cdot 0.5646 \approx 0.3215$ and $y_2 \approx 0.8 \cdot 0.5646 + 0.6 \cdot 0.8253 \approx 0.9469$. One more step: $\cos(0.9469) \approx 0.5847$, $\sin(0.9469) \approx 0.8113$, so $x_3 \approx -0.581$, $y_3 \approx 0.814$. The $x$-values fall, the $y$-values rise then fall. Two more steps give $(x_4, y_4) \approx (-0.9906, 0.1366)$ and $(x_5, y_5) \approx (-0.99999991, 0.00043)$, and the guess $(-1, 0)$ appears.

Now the student must *prove* it, and tries to bound $x_{n+1}$ and $y_{n+1}$ separately. That goes nowhere: each new coordinate is a mixture of both old ones with coefficients $\cos y_n, \sin y_n$ that themselves depend on $y_n$, so there is no monotone quantity in sight and no single-variable recursion. The wall is that the system is two-dimensional and non-linear, and treated coordinate by coordinate it stays that way.

*The model chain.* *Step 1: the map is a rotation.* Write $z_n = x_n + iy_n$. The recursion says exactly

$$z_{n+1} = (x_n + iy_n)(\cos y_n + i \sin y_n) = z_n e^{iy_n},$$

because the real part of the product is $x_n \cos y_n - y_n \sin y_n$ and the imaginary part is $x_n \sin y_n + y_n \cos y_n$. So $(x_{n+1}, y_{n+1})$ is $(x_n, y_n)$ rotated by the angle $y_n$, and $|z_{n+1}| = |z_n|$. Since $0.8^2 + 0.6^2 = 1$, every point lies on the unit circle.

*Step 2: one real variable.* Let $\theta_1 = \cos^{-1}(0.8) \in (0, \pi/2)$, so $(x_1, y_1) = (\cos\theta_1, \sin\theta_1)$. If $(x_n, y_n) = (\cos\theta_n, \sin\theta_n)$ then, by the addition formulas, $(x_{n+1}, y_{n+1}) = (\cos(\theta_n + y_n), \sin(\theta_n + y_n))$. Defining $\theta_{n+1} = \theta_n + y_n$, induction gives $(x_n, y_n) = (\cos\theta_n, \sin\theta_n)$ for all $n$, and

$$\theta_{n+1} = \theta_n + \sin\theta_n.$$

*Step 3: $\theta_n$ stays in $(0, \pi)$ and increases.* For $0 < \theta < \pi$: $\sin\theta > 0$, so $\theta + \sin\theta > \theta > 0$; and $\sin\theta = \sin(\pi - \theta) < \pi - \theta$ because $\sin t < t$ for $t > 0$ (integrate $\cos u \le 1$ from 0 to $t$, with strict inequality since $\cos u < 1$ on $(0, 2\pi)$). Hence $\theta + \sin\theta < \pi$. By induction $0 < \theta_n < \pi$ for all $n$, and $\theta_{n+1} > \theta_n$.

*Step 4: the limit.* An increasing sequence bounded above by $\pi$ converges, to some $L \in (0, \pi]$ (with $L \ge \theta_1 > 0$). Letting $n \to \infty$ in $\theta_{n+1} = \theta_n + \sin\theta_n$ and using continuity of sin gives $\sin L = 0$, so $L = \pi$. By continuity of cos and sin,

$$\lim x_n = \cos\pi = -1, \qquad \lim y_n = \sin\pi = 0. \qquad \square$$

### Layer 2 THE HOW — the bifurcation

**Bifurcating point.** Both chains write down the same two formulas. They split at the first glance at the right-hand side $x_n \cos y_n - y_n \sin y_n$, $x_n \sin y_n + y_n \cos y_n$. The natural attempt reads it as "two coordinates, mixed". The model reads it as "the rotation matrix $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ applied to $(x_n, y_n)$, with $\alpha = y_n$", and from that moment the problem has one variable, not two.

> *$x\cos\alpha - y\sin\alpha$ and $x\sin\alpha + y\cos\alpha$ — that's a rotation by $\alpha$. Here $\alpha$ happens to be $y_n$, but a rotation is a rotation: the length is preserved. And $0.8^2 + 0.6^2 = 1$, so the point lives on the unit circle and the only thing moving is its angle.*

> *On the unit circle $y = \sin\theta$, so "rotate by $y_n$" means "add $\sin\theta_n$ to the angle". I now have a one-dimensional recursion $\theta \mapsto \theta + \sin\theta$, and for one-dimensional recursions I know the drill: find an invariant interval, check monotonicity, pass to the limit in the fixed-point equation.*

- ▷ **Cue:** a pair $(x\cos\alpha - y\sin\alpha,\ x\sin\alpha + y\cos\alpha)$ $\longrightarrow$ **Move:** recognise a rotation; switch to $z = x + iy$ or to polar coordinates, and look for the preserved length
- ▷ **Cue:** a data point like $(0.8, 0.6)$ or $(3/5, 4/5)$ $\longrightarrow$ **Move:** check $x^2 + y^2$ — the numbers were chosen to put you on the unit circle
- ▷ **Cue:** a one-variable recursion $a_{n+1} = f(a_n)$ $\longrightarrow$ **Move:** find an interval $I$ with $f(I) \subseteq I$, show $f(a) - a$ has one sign on $I$, conclude monotone and bounded, then solve $L = f(L)$

**Layer 3 THE WHY**

Why is the rotation not a trick? Because the formula *is* the definition of rotation. The rotation of $(x, y)$ by $\alpha$ is the point $(x\cos\alpha - y\sin\alpha,\ x\sin\alpha + y\cos\alpha)$: that pattern of signs, with the same angle in all four trig functions, appears in exactly one place in A-level mathematics, the matrix of a rotation, and in complex numbers as multiplication by $e^{i\alpha}$. The only disguise is that the angle is one of the coordinates. A disguise of the angle does not change what the map does to lengths: rotations preserve distance from the origin, whatever the angle is.

Then why check $x_1^2 + y_1^2$? Because once you know length is preserved, the length is the invariant, and the first thing to do with an invariant is evaluate it at the start. 0.8 and 0.6 are the 3–4–5 triangle scaled; the examiner chose them to put the point on the unit circle. Had they wanted a general radius they would have written a general starting point.

Kedlaya's solution sets $(x_n, y_n) = (\cos\theta_n, \sin\theta_n)$ and does not say why. Here is where that comes from: on a circle, a point has exactly one degree of freedom, and the natural coordinate for it is the angle. A two-dimensional system confined to a one-dimensional curve should be rewritten in that curve's coordinate. After that, $y_n = \sin\theta_n$ is not a coincidence to be noticed; it is what $y$ means on the unit circle.

Why does $\theta$ climb to $\pi$ and not past it? The fixed points of $\theta \mapsto \theta + \sin\theta$ are the zeros of sin, namely multiples of $\pi$. Between 0 and $\pi$ the step $\sin\theta$ is positive, so $\theta$ moves right; the bound $\sin\theta < \pi - \theta$ says the step is always shorter than the remaining distance to $\pi$, so $\theta$ never jumps over. Geometrically: the point on the upper half circle is rotated anticlockwise by its own height, which is less than the arc still to go to $(-1, 0)$. The convergence is very fast: with $\varphi = \pi - \theta$, the recursion becomes $\varphi_{n+1} = \varphi_n - \sin\varphi_n \approx \varphi_n^3/6$, which is why $y$ drops from 0.137 to 0.00043 in one step and to about $10^{-11}$ in the next.

This is the standard fixed-point analysis for $a_{n+1} = f(a_n)$: invariant interval, sign of $f(a) - a$, monotone convergence, continuity to identify the limit. The problem's only real content is reducing a two-variable system to that template.

**Layer $+\alpha$ BACKTESTING**

**1.** The first cue is the shape $x\cos(\cdot) - y\sin(\cdot)$, $x\sin(\cdot) + y\cos(\cdot)$. On seeing it write $z_{n+1} = z_n e^{iy_n}$ and $x_n^2 + y_n^2 = x_1^2 + y_1^2 = 1$.

**2.** Change to the angle: $(x_n, y_n) = (\cos\theta_n, \sin\theta_n)$, giving $\theta_{n+1} = \theta_n + \sin\theta_n$, then monotone convergence.

**3.** Hesitation comes after the numerics suggest $(-1, 0)$ and the student tries to bound the coordinates separately. Reinforce it away by asking, before any estimate: *what quantity does this map preserve?*

**4.** The trap is the angle being a coordinate, $y_n$, which makes the map look non-linear and hides the rotation. A second trap: concluding from $\sin L = 0$ that $L = 0$ or forgetting to show $\theta_n$ never passes $\pi$, which needs $\sin\theta < \pi - \theta$.

**5.** Whenever a two-variable recursion has the rotation signature, or preserves some quantity, look for the invariant first; it cuts the dimension by one.

**6.** Reusable decision rule: *before estimating a multi-variable iteration, find what it preserves and restrict to that level set; then parametrise the level set by its natural coordinate and treat the result as a one-variable recursion $a_{n+1} = f(a_n)$.*

## 1987 B5

**1987 B5.** Let $O_n$ be the $n$-dimensional vector $(0, 0, \cdots, 0)$. Let $M$ be a $2n \times n$ matrix of complex numbers such that whenever $(z_1, z_2, \ldots, z_{2n})M = O_n$, with complex $z_i$, not all zero, then at least one of the $z_i$ is not real. Prove that for arbitrary real numbers $r_1, r_2, \ldots, r_{2n}$, there are complex numbers $w_1, w_2, \ldots, w_n$ such that

$$\mathrm{re}\left[M\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}\right] = \begin{pmatrix} r_1 \\ \vdots \\ r_{2n} \end{pmatrix}.$$

(Note: if $C$ is a matrix of complex numbers, $\mathrm{re}(C)$ is the matrix whose entries are the real parts of the entries of $C$.)

**Caveat.** The archive's source `putnam_1987_problems.tex` prints the right-hand side as $(r_1, \ldots, r_n)^T$, which is dimensionally impossible ($Mw$ has $2n$ entries). The statement above follows Kedlaya–Poonen–Vakil, *The William Lowell Putnam Mathematical Competition 1985–2000*, which has $r_{2n}$; the problem also says "arbitrary real numbers $r_1, \ldots, r_{2n}$".

**Fields:** linear algebra **Position:** B5 — late in the paper, but short once the right space is chosen **Answer:** Write $M = A + iB$ with $A, B$ real. The hypothesis says the real $2n \times 2n$ matrix $[A\ \ B]$ has no non-zero real left null vector, so it is invertible; and $\mathrm{re}[Mw] = r$ with $w = u + iv$ is the real system $[A\ \ -B]\binom{u}{v} = r$, which is therefore solvable.

### Layer 1 THE WHAT — surface analysis

*The natural attempt.* The student sees "$2n \times n$" and "complex" and thinks about rank. $M$ has $n$ columns, so $Mw$ ranges over a complex subspace of $\mathbb{C}^{2n}$ of dimension at most $n$. The target is a vector in $\mathbb{R}^{2n}$. The instinct is "$n$ unknowns cannot hit $2n$ targets", so the student hunts for where the hypothesis rescues the count.

They try to use the hypothesis directly: it is about left null vectors $z$ with $zM = 0$. Over $\mathbb{C}$, $M$ has rank at most $n$, so the left null space has complex dimension at least $n$ and is certainly non-zero; the hypothesis says none of those vectors is real. The student tries to relate "no real left null vector" to the column space of $M$, perhaps via orthogonal complements in $\mathbb{C}^{2n}$, and runs into conjugates: the natural pairing on $\mathbb{C}^{2n}$ is Hermitian, re is not complex-linear, and every identity picks up bars in awkward places. The wall: the question mixes complex linear algebra (the matrix $M$) with a real-linear operation (re), and working over $\mathbb{C}$ throughout keeps the two at war.

*The model chain.* Everything in the conclusion is real: real targets $r_i$, real parts. So count real dimensions. Write

$$M = A + iB, \qquad w = u + iv,$$

with $A, B$ real $2n \times n$ matrices and $u, v \in \mathbb{R}^n$. Then $Mw = (Au - Bv) + i(Av + Bu)$, so

$$\mathrm{re}[Mw] = Au - Bv = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} u \\ -v \end{pmatrix},$$

where $[A \;\; B]$ is the real $2n \times 2n$ matrix formed by placing $A$ and $B$ side by side. The unknowns are $2n$ real numbers ($u$ and $v$), and the equation is a square real linear system. It is solvable for every $r \in \mathbb{R}^{2n}$ if $[A \;\; B]$ is invertible.

*Invertibility.* A square real matrix is invertible if and only if its rows are linearly independent, i.e. the only real row vector $z$ with $z[A \;\; B] = 0$ is $z = 0$. Suppose $z \in \mathbb{R}^{2n}$ (a row) satisfies $z[A \;\; B] = 0$, i.e. $zA = 0$ and $zB = 0$. Then

$$zM = zA + i\,zB = 0.$$

The hypothesis says any non-zero $z$ with $zM = 0$ has a non-real entry; this $z$ is real, so $z = 0$. Hence $[A \;\; B]$ is invertible.

*Conclusion.* Given $r \in \mathbb{R}^{2n}$, let $\binom{p}{q} = [A \;\; B]^{-1} r$ with $p, q \in \mathbb{R}^n$, and set $u = p$, $v = -q$, $w = u + iv$. Then $\mathrm{re}[Mw] = Au - Bv = Ap + Bq = r$. □

**Remark (the converse holds too).** If some real $z \neq 0$ had $zM = 0$, then $z\,\mathrm{re}[Mw] = \mathrm{re}[zMw] = 0$ for every $w$, so $\mathrm{re}[Mw]$ would be confined to the hyperplane $z^{\perp}$ and could not equal $r = z^T$. So the hypothesis is exactly the condition needed.

**Layer 2 THE HOW — the bifurcation**

**Bifurcating point.** Both chains agree that the question is about the image of $w \mapsto \mathrm{re}[Mw]$. They split at the choice of field. The natural attempt counts over $\mathbb{C}$: $n$ complex unknowns, rank $\leq n$, and gets stuck on the non-linearity of re. The model counts over $\mathbb{R}$: $2n$ real unknowns, $2n$ real equations, one square real matrix.

> *The map $w \mapsto \mathrm{re}[Mw]$ is not complex-linear — re kills that — but it is real-linear. So stop pretending I am over $\mathbb{C}$. Split everything into real and imaginary parts: $n$ complex unknowns are $2n$ real unknowns, and $2n$ real targets. Square system. All I need is that one real $2n \times 2n$ matrix is invertible.*

> *The hypothesis talks about* real *row vectors killing $M$. A real row vector kills $A + iB$ exactly when it kills $A$ and $B$ separately. That is a row vector killing $[A \;\; B]$ — the hypothesis is the invertibility of my square matrix, written in disguise.*

- ▷ **Cue:** a complex matrix together with "real part" or "real numbers" in the conclusion ⟶ **Move:** write $M = A + iB$, $w = u + iv$ and treat the problem as real linear algebra in twice the dimension
- ▷ **Cue:** "whenever $zM = 0$ with $z$ not all zero, then …" ⟶ **Move:** read it as a statement about the left null space, i.e. about linear independence of rows, i.e. invertibility of a square matrix
- ▷ **Cue:** a real vector $z$ and a complex product $zM$ ⟶ **Move:** $zM = 0$ iff $z\,\mathrm{re}M = 0$ and $z\,\mathrm{im}M = 0$, because $z$ has no imaginary part to mix them

**Layer 3 THE WHY**

Why count over $\mathbb{R}$? Because the conclusion is real. The problem asks to hit an arbitrary *real* vector with the *real part* of something. The map $w \mapsto \mathrm{re}[Mw]$ satisfies $\mathrm{re}[M(\lambda w)] = \lambda\,\mathrm{re}[Mw]$ only for real $\lambda$; for $\lambda = i$ it fails. A map is linear over the field whose scalars it respects, and this one respects only $\mathbb{R}$. Linear algebra has to be done over the field of linearity, so the only honest setting is $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$. Once you say that, the dimension count is $2n = 2n$, and the surprise of

the natural attempt ("$n$ unknowns, $2n$ targets") evaporates: a complex unknown is two real unknowns.

Why is the hypothesis stated with row vectors on the left? Because a square matrix $X$ is surjective iff it is injective iff its rows are independent iff $zX = 0 \Rightarrow z = 0$. The problem-setter wrote the invertibility of $[A\ \ B]$ in the form "no non-zero real left null vector" and then disguised $[A\ \ B]$ as $M$ by allowing complex $z$ and saying "at least one is not real". Unwrap the negation: "every non-zero $z$ with $zM = 0$ has a non-real entry" means "no non-zero real $z$ has $zM = 0$". Then the only fact needed is that a real vector multiplies real and imaginary parts separately, so $zM = 0$ splits into $zA = 0$ and $zB = 0$.

Kedlaya's solution opens with "Write $M = A + iB$". Here is where that comes from: the conclusion is real, the hypothesis is about real vectors, and the only complex object is $M$. The move that makes every object in the problem real is to split $M$.

The sign in $\binom{u}{-v}$ is bookkeeping, not an idea: $\mathrm{re}[(A + iB)(u + iv)] = Au - Bv$, and it is cleaner to invert $[A\ \ B]$ and absorb the minus sign into $v$ than to carry $[A\ \ -B]$, though either matrix is invertible exactly when the other is.

The general principle is *realification*: $\mathbb{C}^n$ is $\mathbb{R}^{2n}$, and a complex matrix $A + iB$ acting on $u + iv$ becomes the real block matrix $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$. Taking the real part keeps the top block row $[A\ \ -B]$. This problem is that one line of theory.

**Layer $+\alpha$ BACKTESTING**

**1.** The first cue is "re" in the conclusion next to "arbitrary real numbers". On seeing it write $M = A + iB$, $w = u + iv$, $\mathrm{re}[Mw] = Au - Bv$.

**2.** Real linear algebra in dimension $2n$: reduce to invertibility of the real square matrix $[A\ \ B]$, and test invertibility by left null vectors.

**3.** Hesitation comes from the count "$n$ complex unknowns, $2n$ equations", which feels hopeless. Reinforce it away by always counting real dimensions when any real-part, modulus or conjugate appears.

**4.** The trap is doing complex linear algebra on a map that is only real-linear, and then fighting conjugates in Hermitian orthogonal complements. A smaller trap: misreading the double-negative hypothesis instead of rewriting it as "no real non-zero $z$ with $zM = 0$".

**5.** Any time a problem mixes a complex object with real-valued output (real parts, moduli, real targets), expect to double the dimension and work over $\mathbb{R}$.

**6.** Reusable decision rule: *do linear algebra over the field the map is actually linear over; and turn every "for all targets, a solution exists" into "a square matrix is invertible", tested by "its only left null vector is zero".*

## 1987 B6

**1987 B6.** Let $F$ be the field of $p^2$ elements, where $p$ is an odd prime. Suppose $S$ is a set of $(p^2 - 1)/2$ distinct nonzero elements of $F$ with the property that for each $a \neq 0$ in $F$, exactly one of $a$ and $-a$ is in $S$. Let $N$ be the number of elements in the intersection $S \cap \{2a : a \in S\}$. Prove that $N$ is even.

**Fields:** number theory; finite fields; algebra **Position:** B6 — last problem; hard because the right object (a product) is not suggested by the question, which asks for a count **Answer:** $N$ is always even. Writing $2a = \varepsilon_a s_a$ with $\varepsilon_a = \pm 1$, $s_a \in S$, and multiplying over $S$ gives $2^{(p^2-1)/2} = (-1)^{\#S-N}$ in $F$; the left side is 1 by Fermat, and $\#S$ is even, so $(-1)^N = 1$.

**Layer 1 THE WHAT — surface analysis**

*The natural attempt.* A count whose parity is asked for suggests an involution: pair up the elements of $S \cap 2S$ with no element left alone. The student writes $2S = \{2a : a \in S\}$ and tries

candidate pairings. $\beta \mapsto -\beta$ does not work, since $S$ contains exactly one of $\pm\beta$. $\beta \mapsto 2\beta$ or $\beta \mapsto \beta/2$ do not preserve $S \cap 2S$. Conjugation $\beta \mapsto \beta^p$ (the Frobenius map of $F$ over $\mathbb{F}_p$) is an involution of $F$, but $S$ is arbitrary and need not be closed under it. No pairing respects an arbitrary $S$.

So the student tries to reduce to a nice $S$. Fix a basis $\{1, x\}$ of $F$ over $\mathbb{F}_p$ and take the "half-plane" $S_0 = \{a + bx : a \in H,\ b \in \mathbb{F}_p\} \cup \{bx : b \in H\}$ with $H = \{1, \ldots, \frac{p-1}{2}\}$. Then $S_0 \cap 2S_0$ is $\{a + bx : a \in Q\} \cup \{bx : b \in Q\}$ with $Q = H \cap 2H$, of size $(p+1)\#Q$, which is even. Then show that flipping one element $\alpha \to -\alpha$ changes $N$ by an even number. This works — it is Kedlaya's second solution — but the flip needs a four-case table according to whether $\pm\alpha/2$ and $\pm 2\alpha$ are in $S$, and one must check that the elements $\alpha, 2\alpha, -\alpha, -2\alpha$ are the only ones affected. Correct, careful, long, and it explains nothing about *why* the answer is even.

*The model chain.* Write $2S = \{2a : a \in S\}$ and $\#S = \frac{p^2-1}{2}$.

*Step 1: signs.* For each $a \in S$, the element $2a$ is non-zero (since $2 \neq 0$ in $F$, as $p$ is odd), so exactly one of $2a, -2a$ lies in $S$. Hence there is a unique way to write

$$2a = \varepsilon_a s_a, \qquad \varepsilon_a \in \{1, -1\},\ s_a \in S.$$

By definition $2a \in S$ iff $\varepsilon_a = 1$, so $N = \#\{a \in S : \varepsilon_a = 1\}$ and

$$\prod_{a \in S} \varepsilon_a = (-1)^{\#S - N}.$$

*Step 2: $a \mapsto s_a$ is a bijection of $S$.* If $s_a = s_b$ then $2a = \pm 2b$, so $a = \pm b$; since $S$ contains only one of $b, -b$, $a = b$. An injective map from the finite set $S$ to itself is a bijection.

*Step 3: multiply.* Taking the product of $2a = \varepsilon_a s_a$ over $a \in S$, and using Step 2 to rewrite $\prod s_a = \prod a$,

$$2^{\#S} \prod_{a \in S} a = (-1)^{\#S - N} \prod_{a \in S} a.$$

The product $\prod a$ is non-zero (a product of non-zero elements of a field), so

$$(-1)^{\#S-N} = 2^{\#S} = 2^{(p-1)(p+1)/2} = \left(2^{p-1}\right)^{(p+1)/2} = 1,$$

since $2 \in \mathbb{F}_p \subseteq F$ and $2^{p-1} = 1$ by Fermat's little theorem.

*Step 4: parity.* $p - 1$ and $p + 1$ are both even, so $(p-1)(p+1)$ is divisible by 8 and $\#S = (p^2 - 1)/2$ is even. Thus $(-1)^N = (-1)^{\#S-N} = 1$ in $F$. As $-1 \neq 1$ in $F$ (characteristic $p \neq 2$), $N$ is even. □

*Sanity check, $p = 3$.* In $\mathbb{F}_9$, $2 = -1$, so $2S = -S$ and $S \cap 2S = \varnothing$: $N = 0$. For $p = 5, 7, 11$ a computer check over random $S$ agrees with the theorem.

### Layer 2 THE HOW — the bifurcation

**Bifurcating point.** Both chains start from the same observation: for $a \in S$, either $2a \in S$ or $-2a \in S$. The natural attempt treats this as a membership question and *counts*, trying to pair elements or to track how $N$ changes as $S$ changes. The model treats it as an equation $2a = \pm s_a$ and *multiplies*, converting a count of minus signs into a single power of $-1$.

> *Parity of a count of "yes/no" events is the same as the sign of a product of $\pm 1$'s. Each $a \in S$ gives $2a = \pm$(something in $S$); if I multiply all these equations together, the $\pm$ signs multiply to $(-1)^{\text{number of minus signs}}$, and the "somethings" are $S$ again, so they cancel.*

> *After cancelling I am left with $2^{\#S}$. That is a power of an element of $\mathbb{F}_p$ inside the bigger field, and the exponent $(p^2-1)/2$ is a multiple of $p-1$. Fermat kills it.*

▷ **Cue:** "for each $a$, exactly one of $a$ and $-a$ is in $S$" (a half-system), and a map like $a \mapsto 2a$ $\longrightarrow$ **Move:** write $ka = \varepsilon_a s_a$ with $s_a \in S$ and multiply over $S$ (Gauss's lemma method)

▷ **Cue:** the parity of the number of elements with a property $\longrightarrow$ **Move:** encode as a product of signs $\prod(\pm 1) = (-1)^{\text{count}}$ and compute the product algebraically

▷ **Cue:** a power $c^e$ with $c \in \mathbb{F}_p$ and $(p-1) \mid e$ $\longrightarrow$ **Move:** it equals 1 by Fermat, even inside an extension field

**Layer 3 THE WHY**

Kedlaya's first solution begins "For $a \in S$, there is a unique way to write $2a = \varepsilon_a s_a$". Here is where that comes from.

Start with the definition of $S$. "Exactly one of $a$ and $-a$ is in $S$" says $S$ is a set of representatives for the non-zero elements up to sign: every non-zero $b$ is $\pm s$ for exactly one $s \in S$. So when you apply any map that preserves "non-zero" and commutes with sign — multiplication by 2 is one — each image $2a$ has a well-defined representative $s_a$ and a well-defined sign $\varepsilon_a$. The question "is $2a \in S$?" is exactly the question "is $\varepsilon_a = +1$?". So $N$ is the number of plus signs and $\#S - N$ the number of minus signs.

Now, why multiply? Because parity and signs are the same thing: $(-1)^m$ records $m$ modulo 2 and forgets the rest, which is exactly the information asked for. Products of signs are computable when the objects carrying them can be multiplied, and here they can — they live in a field. The multiplication by 2 permutes the representatives (Step 2), so the product of the $s_a$ is the product of $S$ and cancels; nothing survives except $2^{\#S}$ and the sign. That is the entire reason: the multiplier 2 acts on the $(\#S)$-element set of representatives by a signed permutation, and the product of the signs is forced to equal $2^{\#S}$.

This is Gauss's lemma. Over $\mathbb{F}_p$, with $S = \{1, \ldots, \frac{p-1}{2}\}$, the same computation gives $2^{(p-1)/2} = (-1)^{\#\{\text{minus signs}\}}$, and $2^{(p-1)/2}$ is the Legendre symbol $\left(\frac{2}{p}\right)$ by Euler's criterion. In a general finite field of $q$ elements, the same product gives $(-1)^{\#S-N} = 2^{(q-1)/2}$, which is $+1$ exactly when 2 is a square in $\mathbb{F}_q$. So the invariant fact behind the problem is:

$$N \equiv \#S - (\#S - N) \pmod 2, \qquad (-1)^{\#S-N} = \Big(\text{is 2 a square in } F\Big).$$

In $F = \mathbb{F}_{p^2}$, every element of $\mathbb{F}_p$ is a square (the equation $X^2 = c$ with $c \in \mathbb{F}_p$ has its roots in the quadratic extension, which is $F$); that is the Fermat computation $2^{(p^2-1)/2} = 1$ in disguise. And $\#S = (p^2-1)/2$ is even because $p^2 \equiv 1 \pmod 8$. Both facts are special to $p^2$; over $\mathbb{F}_p$ itself, the analogous $N$ has parity depending on $p$ mod 8, which is why the problem is set in a field of $p^2$ elements.

The count-and-flip approach of the natural attempt is honest but blind to this: it proves the parity is independent of $S$ without identifying what it equals. The product method identifies the invariant as a quadratic character.

**Layer $+\alpha$ BACKTESTING**

**1.** The first cue is the half-system condition "exactly one of $a$ and $-a$ is in $S$" together with the dilation $\{2a : a \in S\}$. On seeing it write $2a = \varepsilon_a s_a$.

**2.** Multiply over $S$ (Gauss's lemma): the $s_a$ run through $S$ and cancel, leaving $\prod \varepsilon_a = 2^{\#S}$.

**3.** Hesitation comes when "prove $N$ is even" pulls you towards searching for a pairing. Reinforce it away by recognising that a parity equals a sign, and signs are computed by products.

**4.** The trap is the combinatorial route: involutions that do not exist for arbitrary $S$, or the reduce-to-a-nice-$S$-then-flip argument, which works but costs a case analysis. Smaller traps: forgetting to prove $a \mapsto s_a$ is a bijection, forgetting that $\#S$ is even, and forgetting that $-1 \neq 1$ needs odd characteristic.

**5.** Any problem with "one of each $\pm$ pair" and a multiplier $k$ is a Gauss's lemma problem; the answer is controlled by whether $k$ is a square.

**6.** Reusable decision rule: *to find the parity of a count, attach a sign* $\pm 1$ *to each item so the count becomes* $(-1)^N$*, then compute the product of signs by multiplying equations in which everything else cancels.*