TMUA9 · TRINITY MATHEMATICS INTERVIEW PREPARATION

ALPHA · OMEGA

Twenty-four Trinity-style admissions quiz mocks

in strictly ascending difficulty, from specimen level to Putnam B6

#		Paper	Tier
1	α	Alpha	Specimen level (entry)
2	β	Beta	Specimen level
3	γ	Gamma	Specimen level
4	δ	Delta	Specimen level
5	ε	Epsilon	Specimen+ (STEP II)
6	ζ	Zeta	Specimen+ (STEP II)
7	η	Eta	STEP III
8	θ	Theta	STEP III
9	ι	Iota	STEP III+
10	κ	Kappa	STEP III+
11	λ	Lambda	Putnam A1/B1
12	μ	Mu	Putnam A1/B1
13	ν	Nu	Putnam A1/B1 to A2/B2
14	ξ	Xi	Putnam A2/B2
15	o	Omicron	Putnam A2/B2 to A3/B3
16	π	Pi	Putnam A3/B3
17	ρ	Rho	Putnam A3/B3 to A4/B4
18	σ	Sigma	Putnam A4/B4
19	τ	Tau	Putnam A4/B4
20	υ	Upsilon	Putnam A4/B4 to A5/B5
21	ϕ	Phi	Putnam A5/B5
22	χ	Chi	Putnam A5/B5
23	ψ	Psi	Putnam A5/B5 to A6/B6
24	Ω	Omega	Putnam B6 (every question)

Format. Each paper follows the Trinity College specimen admissions quizzes: ten questions across algebra, calculus, number theory, combinatorics, geometry, probability and (through Pi) mechanics, with the same instruction rubric. Difficulty varies inside each paper, as in the specimens; the paper average rises monotonically from Alpha to Omega. Questions are original compositions or classical results; none reproduces a Putnam problem from 1965 to 2025. Worked answers are in the separate answer key.

Protocol. Mirror the real quiz: 60 minutes per paper, pick the questions that appeal, write enough to talk through at interview. Then speak each attempted solution aloud for 3 minutes as if to an interviewer.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK α · ALPHA

Trinity-style mock · Paper 1 of 24 · Tier: Specimen level (entry)

There are ten questions below on various areas of mathematics. They are of varying levels of difficulty within this paper: some may be approachable and others hard. You are not expected to answer all of them, or to give complete answers. Attempt those that appeal to you; they will be the basis for discussion in the interview that follows. Bring what you have written with you.

1. Eight people sit around a round table; two seatings are the same if one is a rotation of the other. Two of the people, A and B, refuse to sit next to each other. Show that the number of acceptable seatings is 3600.

2. Show that the line $y = mx + c$ (with $m \neq 0$) is a tangent to the parabola $y^2 = 4ax$ if and only if $c = a/m$. Hence show that the two tangents to the parabola drawn from any point on the line $x = -a$ are perpendicular.

3. Calculate $\displaystyle\int_0^{\pi/2} x\cos^2 x\,dx$.

4. Of the numbers $1, 2, 3, \ldots, 1000$, how many are divisible by none of 2, 5 or 7?

5. Three coins are in a bag. Two are fair and one has heads on both sides. A coin is chosen at random and tossed twice, landing heads both times. What is the probability that it is the double-headed coin? What is the probability that a third toss of the same coin also lands heads?

6. A particle is projected from level ground with speed u at an angle α above the horizontal. Show that its range is $u^2 \sin 2\alpha / g$. The particle is now projected from the foot of a plane inclined at angle β to the horizontal, in the vertical plane containing the line of greatest slope, up the plane. Find the angle of projection (measured from the horizontal) that maximises the range up the plane.

7. Let $\binom{n}{k}$ denote the number of k-element subsets of an n-element set. By counting the ways to choose n people from a group of n men and n women, show that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Using $k\binom{n}{k} = n\binom{n-1}{k-1}$, or otherwise, show that $\displaystyle\sum_{k=0}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$.

[Please turn over]

8. $ABCD$ is a unit square. P is the midpoint of BC and Q is the midpoint of CD. The lines AP and BQ meet at R. Show that AP is perpendicular to BQ and find the ratio $AR : RP$.

9. Sketch the graph of $y = xe^{-x}$. Hence find, for each real number k, the number of real solutions of the equation $xe^{-x} = k$.

10. A sequence is defined by $u_1 = 2$ and $u_{n+1} = \dfrac{1}{2}\left(u_n + \dfrac{2}{u_n}\right)$. Show that

$$u_{n+1} - \sqrt{2} = \frac{(u_n - \sqrt{2})^2}{2u_n}.$$

Deduce that $u_n > \sqrt{2}$ for all n, that (u_n) is decreasing, and that $u_n \to \sqrt{2}$. Roughly how many correct decimal places does u_6 have?

ADMISSIONS QUIZ (MATHEMATICS)

MOCK β · BETA

Trinity-style mock · Paper 2 of 24 · Tier: Specimen level

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1. In how many ways can the letters of MISSISSIPPI be arranged so that no two S's are adjacent?

2. Find all real x satisfying

$$\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1.$$

3. Evaluate $\displaystyle\int_0^{\pi/2} \frac{\sin^{2026} x}{\sin^{2026} x+\cos^{2026} x}\,dx$.

4. Show that n^5-n is divisible by 30 for every integer n. Show further that if n is odd then n^5-n is divisible by 240.

5. Three points are chosen independently and uniformly at random on a circle. Show that the probability that the triangle they form contains the centre of the circle is $\frac{1}{4}$. What is the probability that the triangle is acute?

6. A uniform ladder of length l rests with its foot on rough horizontal ground (coefficient of friction μ) and its top against a smooth vertical wall. Show that the ladder can rest in equilibrium at angle θ to the horizontal if and only if $\tan\theta \geq \dfrac{1}{2\mu}$. Does the answer depend on the mass of the ladder? Explain physically why not.

7. The Fibonacci numbers are defined by $F_0=0$, $F_1=1$, $F_{n+1}=F_n+F_{n-1}$. Show that

$$F_{n+1}F_{n-1}-F_n^2=(-1)^n \qquad (n\geq 1).$$

Deduce that consecutive Fibonacci numbers have no common factor greater than 1.

[Please turn over]

8. A regular hexagon has centre O. What fraction of the area of the hexagon consists of points that are closer to O than to every vertex of the hexagon?

9. Show that, as $n \to \infty$,

$$\left(1+\frac{1}{n}\right)^n = e - \frac{e}{2n} + O\left(\frac{1}{n^2}\right).$$

[Consider $n\ln(1+1/n)$.]

10. Find the maximum value of $x^{1/x}$ for $x > 0$. Hence decide, without a calculator, which is larger: e^π or π^e.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK γ · GAMMA

Trinity-style mock · Paper 3 of 24 · Tier: Specimen level

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1. A path from $(0,0)$ to (n,n) consists of $2n$ unit steps, each either right or up. Show that the number of such paths that never pass strictly above the line $y = x$ is $\dfrac{1}{n+1}\dbinom{2n}{n}$. [Consider what happens when a bad path first touches the line $y = x+1$, and reflect the remainder of the path in that line.]

2. Show that $\tan 3\theta = \dfrac{3t - t^3}{1 - 3t^2}$ where $t = \tan\theta$. Hence show that

$$\tan\frac{\pi}{9}\,\tan\frac{2\pi}{9}\,\tan\frac{4\pi}{9} = \sqrt{3}.$$

3. Let $I_n = \displaystyle\int_0^1 x^n e^x\,dx$ for $n \geq 0$. Show that $I_n = e - nI_{n-1}$ for $n \geq 1$, and that $0 < I_n < \dfrac{e}{n+1}$. Hence show that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{e}.$$

4. Find all pairs of integers (x,y) with $x^2 - y^2 = 2026$. Which positive integers can be written as a difference of two squares of integers?

5. A fair die is rolled until a six appears. What is the probability that the number of rolls needed is even? What is the expected number of rolls needed until two consecutive sixes appear for the first time?

6. A particle is placed at the top of a smooth fixed sphere of radius a and is slightly disturbed. Show that it leaves the surface of the sphere when the radius to the particle makes an angle $\cos^{-1}\frac{2}{3}$ with the upward vertical. How fast is it moving at that moment?

7. Find the remainder when x^{100} is divided by $x^2 - 3x + 2$.

[Please turn over]

8. A triangle has sides of length 13, 14 and 15. Find its area, the radius of its inscribed circle and the radius of its circumscribed circle.

9. Sketch the graph of $y = \dfrac{x^2 - 1}{x^2 - 4}$, showing all asymptotes and turning points, and find the set of values taken by y.

10. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ and that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$. Deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK δ · DELTA

Trinity-style mock · Paper 4 of 24 · Tier: Specimen level

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1. Five married couples are split at random into five pairs, all pairings being equally likely. Show that the probability that nobody is paired with his or her spouse is $\dfrac{544}{945}$.

2. Show that for real θ and positive integer n,

$$\sum_{k=0}^{n} \binom{n}{k} \cos k\theta = \left(2\cos\frac{\theta}{2}\right)^{n} \cos\frac{n\theta}{2}.$$

3. Show that $\displaystyle\int_0^\infty \frac{dx}{1+x^4} = \int_0^\infty \frac{x^2\,dx}{1+x^4}$. By adding the two integrals and substituting $u = x - \dfrac{1}{x}$, evaluate both.

4. Find the last two digits of 7^{2026} and of 3^{27}.

5. A fair coin is tossed repeatedly. Find the expected number of tosses until the pattern HH first appears, and the expected number until HT first appears. Explain in words why these are different, although each pattern has probability $\frac{1}{4}$ at any given position.

6. A uniform flexible chain of length L hangs over a small smooth peg, with lengths a and $L-a$ on the two sides, where $a > L/2$. It is released from rest. Show that at the moment the chain leaves the peg its speed v satisfies $v^2 = \dfrac{2ga(L-a)}{L}$.

7. Show that $\displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{k+1}\binom{n}{k} = \frac{1}{n+1}$, both by integrating $(1-x)^n$ and by a direct algebraic argument.

[Please turn over]

8. A chord of a circle of radius 1 subtends an angle 2θ at the centre and divides the circle into two regions whose areas are in the ratio $1:3$. Show that $2\theta - \sin 2\theta = \dfrac{\pi}{2}$, and estimate θ to two decimal places.

9. Let N be a large number. Explain why the equation $x^x = N$ has exactly one root $w > 1$. Show that w is approximately $\dfrac{\ln N}{\ln \ln N}$, and find an improved approximation.

10. Solve the differential equation $\dfrac{dy}{dx} = (x+y)^2$ with $y(0) = 0$. On what interval does the solution exist?

ADMISSIONS QUIZ (MATHEMATICS)

MOCK ε · EPSILON

Trinity-style mock · Paper 5 of 24 · Tier: Specimen+ (STEP II)

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1. How many subsets of $\{1, 2, \dots, 10\}$ (including the empty set) contain no two consecutive integers? Generalise to $\{1, 2, \dots, n\}$.

2. Show that in any triangle the sum of the squares of the lengths of the three medians equals $\frac{3}{4}$ of the sum of the squares of the lengths of the sides.

3. Evaluate $\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$. [Try $x = \tan\theta$, then $\theta \mapsto \frac{\pi}{4} - \theta$.]

4. Show that there are infinitely many primes of the form $4k + 3$.

5. A stick of length 1 is broken at two points chosen independently and uniformly at random. Show that the probability that the three pieces form a triangle is $\frac{1}{4}$. Now suppose instead that the stick is broken at one uniformly random point, and then the longer of the two pieces is broken at a uniformly random point. Show that the probability of forming a triangle is now $2\ln 2 - 1$.

6. A spherical raindrop falls from rest through a stationary cloud. It starts with negligible radius and gains mass by condensation at a rate proportional to its surface area. Ignoring air resistance, show that its acceleration is constant, and find it.

> Now suppose instead that the drop gains mass at a rate proportional to $\pi r^2 v$ (it sweeps up the droplets in its path). Show that the drop can have constant acceleration $g/7$.

7. Let α, β, γ be the roots of $x^3 - x - 1 = 0$. Find $\alpha^5 + \beta^5 + \gamma^5$ and $\dfrac{1}{1-\alpha} + \dfrac{1}{1-\beta} + \dfrac{1}{1-\gamma}$.

[Please turn over]

8. Find the radius of the sphere inscribed in the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

9. By sketching appropriate graphs, determine how many real solutions the equation $2^x = x^2$ has. Find as many as you can exactly, and locate the others between consecutive integers.

10. The numbers $1, 2, \ldots, 20$ are written on a board. At each step you erase two numbers a and b and write $a + b + ab$ in their place. After 19 steps a single number remains. Show that it does not depend on the order of the steps, and find it.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK ζ · ZETA

Trinity-style mock · Paper 6 of 24 · Tier: Specimen+ (STEP II)

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1. Let a_n be the number of ways to tile a $3 \times 2n$ rectangle with 1×2 dominoes. Show that $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$, with $a_0 = 1$, $a_1 = 3$, and compute a_4.

2. Let a, b, c be positive reals with $a + b + c = 1$. Show that

$$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right) \geq 64.$$

3. Evaluate $\displaystyle\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$.

4. Prove Wilson's theorem: if p is prime then $(p-1)! \equiv -1 \pmod{p}$. Hence find the remainder when $97!$ is divided by 101.

5. A fair coin is tossed repeatedly. Show that the probability that HHH appears before THH is $\frac{1}{8}$. Find the expected number of tosses until HHH first appears, and until HTH first appears.

6. A horizontal platform performs vertical simple harmonic motion with amplitude A and angular frequency ω. A small particle rests on the platform. Show that the particle stays in contact with the platform throughout the motion if and only if $A\omega^2 \leq g$. If $A\omega^2 > g$, at what point of the cycle does the particle first leave the platform?

7. Let P be a polynomial with integer coefficients. Show that $a - b$ divides $P(a) - P(b)$ for all integers $a \neq b$. Deduce that there is no such polynomial with $P(1) = 2$ and $P(3) = 5$. Can there be one with $P(1) = P(2) = P(3) = 1$ and $P(4) = 2$?

[Please turn over]

8. In triangle ABC, points D, E, F lie on BC, CA, AB respectively with $BD : DC = CE : EA = AF : FB = 1 : 2$. Show that the lines AD, BE, CF bound a triangle whose area is $\frac{1}{7}$ of the area of ABC.

9. Let $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Show that $\ln(n+1) < H_n < 1 + \ln n$ for $n \geq 2$. Show that $H_n - \ln n$ is decreasing and bounded below, and that $H_{2n} - H_n \to \ln 2$.

10. Show that $y = \dfrac{\sin x}{x}$ satisfies $xy'' + 2y' + xy = 0$. By substituting $y = u/x$, find the general solution of this differential equation.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK η · ETA

Trinity-style mock · Paper 7 of 24 · Tier: STEP III

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1. A derangement of $\{1, \dots, n\}$ is a permutation with no fixed point. Let D_n be the number of derangements. Show that $D_n = (n-1)(D_{n-1} + D_{n-2})$ and that

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

What happens to the proportion of permutations that are derangements as $n \to \infty$?

2. By factorising $z^{2n} - 1$, or otherwise, show that for every integer $n \geq 2$,

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}.$$

3. Show that $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi$. What does this tell you about π?

4. Show that the only integer solution of $x^2 + y^2 = 3z^2$ is $x = y = z = 0$.

5. A gambler starts with £k and repeatedly bets £1, winning each bet independently with probability p and losing with probability $q = 1 - p$. She stops when she has £0 or £N, where $0 < k < N$. Show that, for $p \neq q$, the probability that she reaches £N is

$$\frac{1 - (q/p)^k}{1 - (q/p)^N}.$$

What is it when $p = q = \tfrac{1}{2}$? What happens as $N \to \infty$ in each case?

6. A uniform rod of length $2a$ stands vertically on a smooth horizontal floor and is slightly disturbed. Explain why its centre of mass moves in a vertical straight line. Find the speed of the centre of mass at the moment the rod becomes horizontal.

7. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y) + xy$ for all real x, y.

[Please turn over]

8. Three circles of radius 1 are each tangent to the other two. Find the radius of the small circle tangent to all three.

9. True or false: some power of 2 has a decimal expansion beginning with the digits 2026.

10. Show that for $n \geq 2$, $\dfrac{1}{n^2} < \dfrac{1}{n-\frac{1}{2}} - \dfrac{1}{n+\frac{1}{2}}$. Deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{5}{3}$.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK θ · THETA

Trinity-style mock · Paper 8 of 24 · Tier: STEP III

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1. Show that the number of ways of writing n as a sum of distinct positive integers (order irrelevant) equals the number of ways of writing n as a sum of odd positive integers (order irrelevant, repetition allowed). List both kinds for $n = 8$.

2. Find all complex z with $z^5 = 1$. By considering $z + z^{-1}$ for a suitable root, show that $\cos\dfrac{2\pi}{5} = \dfrac{\sqrt{5}-1}{4}$.

3. For $a \geq 0$ let $I(a) = \displaystyle\int_0^\infty e^{-ax}\frac{\sin x}{x}\,dx$. Assuming differentiation under the integral sign is valid for $a > 0$, show that $I'(a) = -\dfrac{1}{1+a^2}$. Deduce that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$, stating any further assumption you need.

4. Show that if $n > 1$ is an integer then n does not divide $2^n - 1$.

5. A chord of a circle of radius 1 is chosen "at random". Find the probability that the chord is longer than $\sqrt{3}$ under each of the following models:

(i) its two endpoints are independent and uniform on the circle;

(ii) its distance from the centre is uniform on $[0, 1]$;

(iii) its midpoint is uniform in the disc.

Discuss.

6. A smooth circular hoop of radius a rotates with constant angular velocity ω about a fixed vertical diameter. A bead slides freely on the hoop. Find the positions in which the bead can remain at rest relative to the hoop, and show that a position other than the top and bottom exists if and only if $\omega^2 > g/a$. Which equilibria are stable?

7. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ have real coefficients and only real roots. Show that if $a_{n-1} = a_{n-2} = 0$ then every root of P is 0.

[Please turn over]

8. Show that for any point inside an equilateral triangle, the sum of its perpendicular distances to the three sides is the same. State and prove the analogue for a regular tetrahedron. For a point inside a square, is the sum of the distances to the four vertices minimised at the centre?

9. Show that for each $n \geq 2$ the equation $x^n + x = 1$ has exactly one root x_n in $(0, 1)$, and that $x_n \to 1$. Show that $1 - x_n$ is approximately $\dfrac{\ln n - \ln \ln n}{n}$ for large n.

10. With $F_0 = 0$, $F_1 = 1$, $F_{k+1} = F_k + F_{k-1}$, show that $\displaystyle\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n}$.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK ι · IOTA

Trinity-style mock · Paper 9 of 24 · Tier: STEP III+

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1. Show that among any $n+1$ numbers chosen from $\{1, 2, \ldots, 2n\}$ there are two, one of which divides the other. Show that this is false for n numbers.

2. Find all real x such that $x^3 - 3x = \sqrt{x+2}$. [Try $x = 2\cos\theta$.]

3. Show that $\displaystyle\int_0^1 x^{-x}\,dx = \sum_{n=1}^{\infty} n^{-n}$.

4. Show that the product of four consecutive positive integers is never a perfect square.

5. At a party n people leave their hats in a pile, and each later takes a hat uniformly at random (all $n!$ assignments equally likely). Let X be the number of people who get their own hat. Show that $\mathbb{E}[X] = 1$ and $\mathrm{Var}(X) = 1$ for $n \geq 2$.

6. A particle is projected from the origin with fixed speed u in any direction in a vertical plane. Show that the points it can reach are exactly those on or below the parabola

$$y = \frac{u^2}{2g} - \frac{gx^2}{2u^2}.$$

7. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all real x, y.

[Please turn over]

8. Show that there is no equilateral triangle in the plane all of whose vertices have integer coordinates. Is there one in three-dimensional space?

9. Find the least real number c such that $\cosh x \leq e^{cx^2}$ for all real x.

10. Let $a_1 = 1$ and $a_{n+1} = a_n + \dfrac{1}{a_n}$. Show that $a_n^2 \geq 2n - 1$ for all n and that $44 < a_{1000} < 45$.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK κ · KAPPA

Trinity-style mock · Paper 10 of 24 · Tier: STEP III+

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1. Let $S = \{1, 2, \ldots, n\}$. Show that $\sum_{A \subseteq S} |A| = n2^{n-1}$. Evaluate $\sum_{A,B \subseteq S} |A \cap B|$ and $\sum_{A,B \subseteq S} |A \cup B|$, the sums being over all ordered pairs of subsets.

2. Show that $x - \frac{x^2}{2} < \ln(1+x) < x$ for $x > 0$. Hence find

$$\lim_{n \to \infty} \prod_{k=1}^{n} \left(1 + \frac{k}{n^2}\right).$$

3. Show that $\displaystyle\int_0^{\pi/2} \ln(\sin x)\, dx = -\frac{\pi}{2} \ln 2$.
[Compare it with $\int_0^{\pi/2} \ln(\cos x)\, dx$ and $\int_0^{\pi} \ln(\sin x)\, dx$.]

4. Let $F_n = 2^{2^n} + 1$. Show that $F_0 F_1 \cdots F_{n-1} = F_n - 2$. Deduce that distinct F_m, F_n have no common factor greater than 1, and hence that there are infinitely many primes.

5. A needle of length l is dropped at random onto a floor ruled with parallel lines a distance d apart, where $l \leq d$. Show that the probability that the needle crosses a line is $\dfrac{2l}{\pi d}$. How could you use this to estimate π?

6. Masses m and $3m$ hang from the ends of a light inextensible string passing over a smooth light pulley B. The pulley B hangs from one end of a second light string passing over a fixed smooth pulley A; the other end carries a mass M. Find M such that M remains at rest when the system is released. Explain why M is not $4m$.

7. Evaluate $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{2^k}$ and $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{2^k}$.

[Please turn over]

8. A quadrilateral has sides of length $1, 2, 3, 4$ in that order. Show that its area is at most $2\sqrt{6}$, and describe the quadrilateral that achieves this.

9. True or false?

(i) Every positive integer is a sum of distinct Fibonacci numbers, no two of which are consecutive Fibonacci numbers.

(ii) Every integer (positive, negative or zero) can be written as $\sum_{k} \varepsilon_k 3^k$ with each $\varepsilon_k \in \{-1, 0, 1\}$, in exactly one way.

10. Find the general solution of $y'' + y = \sec x$ on $(-\pi/2, \pi/2)$.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK λ · LAMBDA

Trinity-style mock · Paper 11 of 24 · Tier: Putnam A1/B1

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1. A composition of n is an ordered sum of positive integers equal to n. Show that the number of compositions of n with every part equal to 1 or 2 is F_{n+1}, and that the number with every part odd is F_n, where $F_1 = F_2 = 1$.

2. Show that $x^4 + 4y^4 = (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy)$. Deduce that $n^4 + 4^n$ is composite for every integer $n > 1$.

3. Evaluate $\displaystyle\int_0^1 \frac{x-1}{\ln x}\,dx$. [Consider $I(t) = \int_0^1 \frac{x^t-1}{\ln x}\,dx$.]

4. Show that $\lfloor (2+\sqrt{3})^n \rfloor$ is odd for every positive integer n.

5. A and B take turns rolling a fair die, A first. A wins if she rolls a 6; B wins if he rolls a 5 or a 6. The first to win ends the game. Who is more likely to win, and with what probability?

6. A uniform disc and a uniform hoop are released from rest on a rough plane inclined at α to the horizontal and roll without slipping. Find the acceleration of each, and the least coefficient of friction required for the disc to roll without slipping.

7. Find all polynomials P with real coefficients such that $P(x^2) = P(x)P(x+1)$ for all real x.

[Please turn over]

8. In triangle ABC, $AB = 5$, $AC = 3$ and $\angle BAC = 120^\circ$. Find the length of the bisector of angle A from A to BC.

9. Without a calculator, decide which is larger: 2^{100} or 3^{63}.

10. Let $x_0 = \frac{5}{2}$ and $x_{n+1} = x_n^2 - 2$. Find a closed form for x_n.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK μ · MU

Trinity-style mock · Paper 12 of 24 · Tier: Putnam A1/B1

There are ten questions below on various areas of mathematics. They are of varying levels of difficulty within this paper: some may be approachable and others hard. You are not expected to answer all of them, or to give complete answers. Attempt those that appeal to you; they will be the basis for discussion in the interview that follows. Bring what you have written with you.

1. Show that in any group of six people there are either three mutual acquaintances or three mutual strangers. Show that this can fail for five people.

2. Show that for positive reals a, b, c,

$$a^a b^b c^c \geq (abc)^{(a+b+c)/3}.$$

3. For $a > 0$ evaluate $\displaystyle\int_0^\infty \frac{\ln x}{x^2 + a^2}\, dx$.

4. Show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is never an integer for $n \geq 2$.

5. A particle performs a random walk on the integers, starting at 0 and moving ± 1 with probability $\frac{1}{2}$ each at every step. Show that the probability u_{2n} that it is at 0 after $2n$ steps is $\binom{2n}{n} 4^{-n}$, and that $u_{2n} \geq \dfrac{1}{2\sqrt{n}}$. Deduce that the expected number of returns to 0 is infinite.

6. A particle of mass m is attached to one end of a light elastic string of natural length l and modulus of elasticity $2mg$, the other end being fixed at a point O. The particle is released from rest at O. Find the greatest distance below O that the particle reaches.

7. Let P be a polynomial with real coefficients such that $P(x) \geq 0$ for all real x. Show that there are polynomials A, B with real coefficients such that $P = A^2 + B^2$.

[Please turn over]

8. Find the largest possible area of a triangle inscribed in the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

9. True or false: for every positive integer n there is a positive multiple of n whose decimal digits are all 0s and 1s.

10. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{H_n}{2^n}$, where $H_n = 1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n}$.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK ν · NU

Trinity-style mock · Paper 13 of 24 · Tier: Putnam A1/B1 to A2/B2

There are ten questions below on various areas of mathematics. They are of varying levels of difficulty within this paper: some may be approachable and others hard. You are not expected to answer all of them, or to give complete answers. Attempt those that appeal to you; they will be the basis for discussion in the interview that follows. Bring what you have written with you.

1. Two opposite corner squares are removed from an 8×8 chessboard. Show that the rest cannot be tiled by 1×2 dominoes. Now show that if any one white square and any one black square are removed, the rest can always be tiled.

2. Let $f : [0,1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$. Show that for every positive integer n there is $x \in [0, 1 - \frac{1}{n}]$ with $f(x) = f(x + \frac{1}{n})$. Show that the corresponding statement is false for the length $\frac{2}{5}$ in place of $\frac{1}{n}$.

3. For real a with $|a| \neq 1$, evaluate $\displaystyle\int_0^{\pi} \ln\left(1 - 2a\cos x + a^2\right) dx$.

4. Show that for every positive integer n there are n consecutive positive integers none of which is a prime or a power of a prime.

5. $n \geq 2$ points are chosen independently and uniformly on a circle. Show that the probability that all of them lie in some semicircle is $\dfrac{n}{2^{n-1}}$.

6. A rabbit runs along a straight line at constant speed v. A dog, starting at distance d from the rabbit's starting point in the direction perpendicular to the rabbit's path, runs at constant speed $w > v$, always heading straight for the rabbit. Show that the dog catches the rabbit after time $\dfrac{dw}{w^2 - v^2}$.

7. Show that there are no real $n \times n$ matrices A, B with $AB - BA = I$. Show also that if $A^2 = A$ then $\operatorname{rank} A = \operatorname{tr} A$.

[Please turn over]

8. Five points in the plane are given, no three collinear. Show that four of them are the vertices of a convex quadrilateral.

9. Show that for all real numbers $x_1, \dots, x_n$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \min(i, j)\, x_i x_j \geq 0.$$

10. Let $x_0 \in (0, \pi)$ and $x_{n+1} = \sin x_n$. Show that $x_n \to 0$ and that $\sqrt{n}\, x_n \to \sqrt{3}$.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK ξ · XI

Trinity-style mock · Paper 14 of 24 · Tier: Putnam A2/B2

There are ten questions below on various areas of mathematics. They are of varying levels of difficulty within this paper: some may be approachable and others hard. You are not expected to answer all of them, or to give complete answers. Attempt those that appeal to you; they will be the basis for discussion in the interview that follows. Bring what you have written with you.

1. A descent of a permutation σ of $\{1,\dots,n\}$ is an index $i<n$ with $\sigma(i)>\sigma(i+1)$. Show that the number of permutations with exactly one descent is 2^n-n-1.

2. Let $f:[0,1]\to\mathbb{R}$ be continuous with $\int_0^1 f(x)x^n\,dx=0$ for every $n\ge 0$. Show that $f\equiv 0$. (You may quote the Weierstrass approximation theorem.)

3. Evaluate $\displaystyle\int_0^1 \ln x\,\ln(1-x)\,dx$. (You may use $\sum_{n\ge1} n^{-2}=\pi^2/6$.)

4. Find all positive integers n such that $\varphi(n)$ divides n, where φ is Euler's totient function.

5. An urn contains one red and one blue ball. Repeatedly, a ball is drawn at random and returned together with another ball of the same colour. Show that after n draws the number of red balls in the urn is uniformly distributed on $\{1,2,\dots,n+1\}$.

6. Snow starts falling at some time before noon, at a constant rate. A snowplough starts at noon; its speed is inversely proportional to the depth of snow. It travels 2 km in the first hour and 1 km in the second hour. When did the snow start falling?

7. Let A be a real $n\times n$ matrix with $A^3=A+I$. Show that $\det A>0$.

[Please turn over]

8. For a triangle with all angles less than 120°, the point P minimising $PA + PB + PC$ is the point where each side subtends 120°. Prove this, and find the minimum value of $PA + PB + PC$ for the triangle with sides $3, 4, 5$.

9. Show that for positive reals a, b, c,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Show that $\dfrac{a}{a+b} + \dfrac{b}{b+c} + \dfrac{c}{c+a}$ always lies strictly between 1 and 2, and that these bounds cannot be improved.

10. Show that $\cot^2 x < \dfrac{1}{x^2} < 1 + \cot^2 x$ for $0 < x < \frac{\pi}{2}$, and that $\displaystyle\sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1} = \frac{m(2m-1)}{3}$.

Deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK *o* · OMICRON

Trinity-style mock · Paper 15 of 24 · Tier: Putnam A2/B2 to A3/B3

There are ten questions below on various areas of mathematics. They are of varying levels of difficulty within this paper: some may be approachable and others hard. You are not expected to answer all of them, or to give complete answers. Attempt those that appeal to you; they will be the basis for discussion in the interview that follows. Bring what you have written with you.

1. Show that any sequence of n^2+1 distinct real numbers has a monotonic subsequence of length $n+1$. Show that n^2 terms do not suffice.

2. Show that there is no continuous function $f:\mathbb{R}\to\mathbb{R}$ with $f(f(x))=-x$ for all x. Is there a (discontinuous) one?

3. Evaluate $\displaystyle\int_0^{\pi/2} \frac{x}{\tan x}\,dx$.

4. Show that $x^4+y^4=z^2$ has no solutions in positive integers.

5. A stick of length 1 is broken into n pieces at $n-1$ independent uniformly random points. Show that the expected length of the shortest piece is $\dfrac{1}{n^2}$. What is the expected length of the longest piece?

6. A uniform heavy flexible chain hangs in equilibrium between two points at the same height. Show that its shape is $y=c\cosh(x/c)$ for suitable axes and a constant c.

7. Show that the $n\times n$ matrix H with entries $H_{ij}=\dfrac{1}{i+j-1}$ is positive definite, that is, $x^{T}Hx>0$ for every nonzero real vector x.

[Please turn over]

8. Show that every convex polygon of area 1 is contained in some rectangle of area 2.

9. Show that $\left(1+\frac{1}{n}\right)^{n+1}$ is decreasing in n and $\left(1+\frac{1}{n}\right)^{n}$ is increasing.

10. Does $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n}$ converge? Does $\displaystyle\sum_{n=1}^{\infty}\frac{|\sin n|}{n}$?

ADMISSIONS QUIZ (MATHEMATICS)

MOCK π · PI

Trinity-style mock · Paper 16 of 24 · Tier: Putnam A3/B3

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1. A family $\mathcal{F}$ of subsets of $\{1,\dots,n\}$ is an antichain if no member contains another. Show that

$$\sum_{A\in\mathcal{F}} \binom{n}{|A|}^{-1} \le 1,$$

and deduce that $|\mathcal{F}| \le \binom{n}{\lfloor n/2\rfloor}$.

2. Let $f:[0,1]\to\mathbb{R}$ be differentiable with $f(0)=0$ and $f(1)=1$. Show that there are distinct $a,b\in(0,1)$ with $f'(a)f'(b)=1$.

3. Let f be continuous on $[0,1]$. Show that $\lim_{n\to\infty} n\int_0^1 x^n f(x)\,dx = f(1)$. Find the limit of $n\displaystyle\int_0^1 \frac{x^n}{1+x}\,dx$, and then the next term in the asymptotic expansion of $\displaystyle\int_0^1 \frac{x^n}{1+x}\,dx$.

4. Let $p>3$ be prime and write $1+\frac12+\cdots+\frac{1}{p-1}=\frac{a}{b}$ in lowest terms. Show that p^2 divides a.

5. n candidates for a job are interviewed in a uniformly random order. After each interview you must accept or reject immediately; you can rank each candidate only relative to those already seen. You reject the first r and then accept the first candidate better than all previous ones. Find the probability of choosing the best candidate, and show that the optimal r is about n/e, with success probability about $1/e$.

6. A ball is dropped from height h onto a horizontal floor; the coefficient of restitution is $e<1$. Show that it bounces infinitely often but comes to rest after a finite time, and find that time and the total distance travelled.

7. Show that the $n\times n$ matrix with (i,j) entry $\gcd(i,j)$ has determinant $\varphi(1)\varphi(2)\cdots\varphi(n)$. [Write $\gcd(i,j)=\sum_{d\mid i,\,d\mid j}\varphi(d)$.]

[Please turn over]

8. Let O and I be the circumcentre and incentre of a triangle with circumradius R and inradius r. Show that $OI^2 = R(R-2r)$. Deduce that $R \geq 2r$, with equality exactly for equilateral triangles.

9. Let $a_n \geq 0$ with $\sum a_n < \infty$. Show that

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \leq e \sum_{n=1}^{\infty} a_n.$$

[Apply AM-GM to the numbers ka_k and use $(n!)^{1/n} > (n+1)/e$.]

10. Evaluate $\displaystyle\sum_{n=0}^{\infty} \binom{2n}{n}^{-1}$.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK ρ · RHO

Trinity-style mock · Paper 17 of 24 · Tier: Putnam A3/B3 to A4/B4

There are ten questions below on various areas of mathematics. They are of varying levels of difficulty within this paper: some may be approachable and others hard. You are not expected to answer all of them, or to give complete answers. Attempt those that appeal to you; they will be the basis for discussion in the interview that follows. Bring what you have written with you.

1. Show that the number of labelled trees on n vertices is n^{n-2}.

2. Show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ that takes every real value exactly twice. Is there one that takes every real value exactly three times?

3. Show that $\displaystyle\int_0^1\!\!\int_0^1 \frac{dx\,dy}{1-xy} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, and that $\displaystyle\int_0^{\infty} \frac{x^3}{e^x-1}\,dx = 6\sum_{n=1}^{\infty}\frac{1}{n^4}$.

4. Let $p \equiv 1 \pmod 4$ be prime. Show that there is an integer a with $a^2 \equiv -1 \pmod p$. By considering the numbers $u + av$ with $0 \le u, v < \sqrt{p}$, show that p is a sum of two squares.

5. A permutation of $\{1, \dots, n\}$ is chosen uniformly at random. Show that the probability that 1 and 2 lie in the same cycle is $\frac{1}{2}$ (for $n \ge 2$), and that the expected number of cycles is $1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

6. A simple pendulum of length l swings with amplitude α. Show that its period is

$$T = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}, \qquad k = \sin\frac{\alpha}{2}.$$

Deduce that the period increases with amplitude and tends to infinity as $\alpha \to \pi$. Find the first correction term to $2\pi\sqrt{l/g}$ for small α.

7. Show that for real $n \times n$ matrices A, B, the matrices AB and BA have the same characteristic polynomial.

[Please turn over]

8. A finite set of points in the plane is not contained in a line. Show that there is a line passing through exactly two of the points.

9. Let $x_1, \dots, x_n \in [0,1]$. Show that $\displaystyle\sum_{1 \le i < j \le n} |x_i - x_j| \le \left\lfloor \frac{n^2}{4} \right\rfloor$, and find when equality holds.

10. Show that $\ln n! - \left(n + \frac{1}{2}\right)\ln n + n$ is decreasing and converges to a limit c. Using Wallis's product, show that $e^c = \sqrt{2\pi}$, so that $n! \sim \sqrt{2\pi n}\,(n/e)^n$.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK σ · SIGMA

Trinity-style mock · Paper 18 of 24 · Tier: Putnam A4/B4

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1. Each edge of the complete graph on 17 vertices is coloured red, green or blue. Show that there is a triangle with all three edges the same colour.

2. Let $f:[0,1]\to\mathbb{R}$ be continuous with $\displaystyle\int_0^1 f(x)\,dx=\int_0^1 xf(x)\,dx=0$. Show that f has at least two zeros in $(0,1)$. Generalise.

3. Show that $\displaystyle\int_0^\infty\left(\frac{\sin x}{x}\right)^2dx=\frac{\pi}{2}$ and $\displaystyle\int_0^\infty\left(\frac{\sin x}{x}\right)^3dx=\frac{3\pi}{8}$. (You may use $\int_0^\infty\frac{\sin x}{x}\,dx=\frac{\pi}{2}$.)

4. Show that a positive integer is a sum of two squares if and only if every prime $q\equiv 3 \pmod 4$ divides it to an even power. (You may assume every prime $p\equiv 1 \pmod 4$ is a sum of two squares.)

5. In an election A receives a votes and B receives b votes, with $a>b$. The votes are counted in a uniformly random order. Show that the probability that A is strictly ahead throughout the count is $\dfrac{a-b}{a+b}$.

6. A particle moves under a central force $F(r)$ directed towards the origin. Show that it moves in a plane and sweeps out area at a constant rate. Writing $u=1/r$ and $h=r^2\dot\theta$, show that

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{mh^2u^2},$$

and deduce that an inverse-square force gives an orbit that is a conic section.

7. Show that $\det\left(a_i^{j-1}\right)_{i,j=1}^{n}=\prod_{i<j}(a_j-a_i)$. Deduce that for distinct reals $a_1,\dots,a_n$ the functions $e^{a_1x},\dots,e^{a_nx}$ are linearly independent.

[Please turn over]

8. Show that for any four points A, B, C, D in the plane,

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD,$$

with equality if and only if A, B, C, D lie in this order on a circle or a line.

9. Let $a_n \geq 0$ and $A_n = a_1 + \cdots + a_n$. Show that

$$\sum_{n=1}^{\infty} \left(\frac{A_n}{n} \right)^2 \leq 4 \sum_{n=1}^{\infty} a_n^2.$$

10. Let $a_0 = 1$ and $a_{n+1} = a_n + e^{-a_n}$. Show that $a_n - \ln n \to 0$.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK τ · TAU

Trinity-style mock · Paper 19 of 24 · Tier: Putnam A4/B4

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1. Show that every k-regular bipartite graph ($k \geq 1$) has a perfect matching. Deduce that if a standard pack of 52 cards is dealt into 13 piles of 4, one card can be chosen from each pile so that the 13 chosen cards contain exactly one card of each rank.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with $f(x) + f'(x) \to L$ as $x \to \infty$. Show that $f(x) \to L$. Is the same true if $f(x) + 2f'(x) \to L$? If $f(x) - f'(x) \to L$?

3. Show that $\displaystyle\int_0^1\!\!\int_0^1 \frac{dx\,dy}{1-x^2y^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$. Using the substitution $x = \dfrac{\sin u}{\cos v}$, $y = \dfrac{\sin v}{\cos u}$, show that the integral equals $\dfrac{\pi^2}{8}$, and deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

4. Show that for every positive integer n there is an n-digit number, all of whose digits are odd, that is divisible by 5^n.

5. A symmetric simple random walk on $\mathbb{Z}$ starts at 0. Show that the expected time until it first reaches $+n$ or $-n$ is n^2. Show that with probability 1 it eventually reaches $+1$, but that the expected time to do so is infinite.

6. Show that every closed bounded convex set in the plane has a circumscribed square: a square each of whose four sides meets the set, with the set inside the square.

7. Let A be a real $n \times n$ matrix with $A^T = -A$. Show that $\det(I + A) \geq 1$.

[Please turn over]

8. Show that a convex polygon has at most three acute interior angles. Show that a (non-convex) simple quadrilateral can have three acute angles, and investigate the maximum number of acute angles of a simple n-gon.

9. Let $x_1, \ldots, x_n$ be positive with $x_1 + \cdots + x_n = 1$, where $n \geq 2$. Show that

$$\sum_{i=1}^{n} \frac{x_i}{\sqrt{1 - x_i}} \geq \sqrt{\frac{n}{n-1}}.$$

10. Evaluate $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\left\lfloor \frac{2n}{k} \right\rfloor - 2 \left\lfloor \frac{n}{k} \right\rfloor \right)$.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK υ · UPSILON

Trinity-style mock · Paper 20 of 24 · Tier: Putnam A4/B4 to A5/B5

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1. Let p be prime. Show that among any $2p-1$ integers there are p whose sum is divisible by p. [Count, modulo p, the solutions in $\mathbb{F}_p^{2p-1}$ of the system $\sum a_i x_i^{p-1} = 0$, $\sum x_i^{p-1} = 0$.]

2. Let $a_n > 0$ with $\sum a_n$ convergent. Show that $\sum a_n^{1-1/n}$ converges.

3. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}} = \frac{\pi^2}{18}$. [Consider the Maclaurin series of $(\arcsin x)^2$.]

4. Show that $\sum_{d \mid n} \varphi(d) = n$ for every positive integer n. Deduce that for every prime p there is an integer g whose powers $g, g^2, \dots, g^{p-1}$ are distinct modulo p.

5. Three points are chosen independently and uniformly in the unit square. Show that the expected area of the triangle they form is $\dfrac{11}{144}$.

6. Let f be a continuously differentiable 2π-periodic function with $\int_0^{2\pi} f = 0$. Show that $\int_0^{2\pi} f^2 \le \int_0^{2\pi} (f')^2$ (Wirtinger). Using this, show that a smooth closed curve of length L enclosing area A satisfies $L^2 \ge 4\pi A$, with equality only for circles.

7. Let A be a real $n \times n$ matrix with every diagonal entry 0 and every off-diagonal entry ± 1. Show that if n is even then A is invertible.

[Please turn over]

8. A rectangle is tiled by finitely many smaller rectangles with sides parallel to its own. Each small rectangle has at least one side of integer length. Show that the big rectangle has a side of integer length. [Consider $\iint e^{2\pi i(x+y)}\,dx\,dy$.]

9. Let $a, b, c > 0$ with $a + b + c = 3$. Show that $\sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca$.

10. Evaluate $\displaystyle\lim_{n\to\infty} \frac{(n!)^{1/n}}{n}$ and $\displaystyle\lim_{n\to\infty} \left(\frac{(2n)!}{n!\,n^n}\right)^{1/n}$.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK ϕ · PHI

Trinity-style mock · Paper 21 of 24 · Tier: Putnam A5/B5

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1. The edges of the complete graph K_n are partitioned into the edge sets of complete bipartite graphs. Show that at least $n-1$ complete bipartite graphs are needed. [Associate to each one a quadratic form.]

2. Let $f:[0,1]\to[0,1]$ be continuous, $x_0\in[0,1]$ and $x_{n+1}=f(x_n)$. Show that (x_n) converges if and only if $x_{n+1}-x_n\to 0$.

3. Show that for $0<s<1$,
$$\int_0^\infty \frac{x^{s-1}}{1+x}\,dx=\frac{\pi}{\sin \pi s}.$$

4. Let $n\geq 2$. Using the cyclotomic polynomial Φ_n, show that every prime factor p of $\Phi_n(a)$ with $p\nmid n$ satisfies $p\equiv 1 \pmod n$. Deduce that there are infinitely many primes $\equiv 1 \pmod n$.

5. Let P_N be the probability that two integers chosen independently and uniformly from $\{1,\dots,N\}$ are coprime. Show that $P_N\to 6/\pi^2$.

6. Let $n\geq 3$. Show that if a regular n-gon in $\mathbb{R}^k$ (for any k) has all its vertices at integer points then $n\in\{3,4,6\}$. Show that each of $3,4,6$ is achieved in $\mathbb{R}^3$, and that only 4 is achieved in $\mathbb{R}^2$.

7. Show that the $n\times n$ matrix with (i,j) entry $\binom{i+j}{i}$, for $0\leq i,j\leq n-1$, has determinant 1.

[Please turn over]

8. A disc of radius 1 is covered by finitely many parallel-sided strips (planks). Show that the sum of the widths of the strips is at least 2. [Lift the picture to a sphere and use Archimedes' hat-box theorem.]

9. Let $0 < m \le a_i \le M$ for $i = 1, \dots, n$. Show that

$$\left(\sum a_i\right)\left(\sum \frac{1}{a_i}\right) \le \frac{n^2(m+M)^2}{4mM},$$

and determine when equality is possible.

10. Show that $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^n = \frac{e}{e-1}$, and find the constant C such that

$$\sum_{k=1}^{n} \left(\frac{k}{n}\right)^n = \frac{e}{e-1} - \frac{C}{n} + O\left(\frac{1}{n^2}\right).$$

ADMISSIONS QUIZ (MATHEMATICS)

MOCK χ · CHI

Trinity-style mock · Paper 22 of 24 · Tier: Putnam A5/B5

There are ten questions below on various areas of mathematics. They are of varying levels of difficulty within this paper: some may be approachable and others hard. You are not expected to answer all of them, or to give complete answers. Attempt those that appeal to you; they will be the basis for discussion in the interview that follows. Bring what you have written with you.

1. A tournament is a complete graph in which every edge is given a direction. Show that every tournament has a directed path through all its vertices. Show that if a tournament is strongly connected (every vertex can reach every other) then it has a directed cycle through all its vertices.

2. Show that for real $a_1, a_2, \ldots$ with $\sum a_n^2 < \infty$,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n} \le \pi \sum_{n=1}^{\infty} a_n^2,$$

and that the constant π cannot be replaced by anything smaller.

3. Evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{n}$, where $H_n = 1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n}$.

4. Show that $\displaystyle\sum_{p \text{ prime}}\frac{1}{p}$ diverges.

5. A rigid wire of length L, bent into any (piecewise smooth) plane shape, is dropped at random onto a floor ruled with parallel lines a distance d apart. Show that the expected number of crossings is $\dfrac{2L}{\pi d}$, regardless of the shape. Deduce that every plane convex curve of constant width w has perimeter πw.

6. Show that for every n there are n points in the plane, no three collinear, with all pairwise distances integers.

7. Let A be an $n \times n$ matrix with every entry ± 1. Show that $|\det A| \le n^{n/2}$. Show that if equality holds and $n > 2$ then $4 \mid n$.

[Please turn over]

8. Show that for all real $x_1, \dots, x_n$,

$$\sum_{i,j} |x_i - x_j| \le \sum_{i,j} |x_i + x_j|.$$

[One route: $|a| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos at}{t^2}\,dt$.]

9. Show that if $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ then the edges of K_n can be coloured red and blue with no monochromatic K_k. Deduce that such a colouring exists for $n = \lfloor 2^{k/2} \rfloor$ when $k \ge 3$.

10. Let $s(k)$ be the number of 1s in the binary expansion of k. Show that for $0 \le j < m$,

$$\sum_{k=0}^{2^m-1} (-1)^{s(k)} k^j = 0.$$

Deduce a partition of $\{0, 1, \dots, 15\}$ into two sets of eight with equal sums, equal sums of squares and equal sums of cubes.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK ψ · PSI

Trinity-style mock · Paper 23 of 24 · Tier: Putnam A5/B5 to A6/B6

There are ten questions below on various areas of mathematics. They are of varying levels of difficulty within this paper: some may be approachable and others hard. You are not expected to answer all of them, or to give complete answers. Attempt those that appeal to you; they will be the basis for discussion in the interview that follows. Bring what you have written with you.

1. Let $n \geq 2k$ and let $\mathcal{F}$ be a family of k-element subsets of $\{1, \dots, n\}$, any two of which intersect. Show that $|\mathcal{F}| \leq \binom{n-1}{k-1}$. [Arrange $1, \dots, n$ around a circle and count arcs.]

2. Show that if f is differentiable on $[a, b]$ then f' takes every value between $f'(a)$ and $f'(b)$, even though f' need not be continuous. Give a differentiable function on $\mathbb{R}$ whose derivative is discontinuous.

3. Show that

$$\int_0^1 \frac{\arctan\sqrt{2+x^2}}{(1+x^2)\sqrt{2+x^2}}\,dx = \frac{5\pi^2}{96}.$$

4. Let p be an odd prime. Show that the number of pairs (x, y) of residues modulo p with $x^2 + y^2 \equiv 1 \pmod{p}$ is $p - (-1)^{(p-1)/2}$.

5. A permutation of $\{1, \dots, n\}$ is chosen uniformly at random. Show that for $k > n/2$ the probability that it has a cycle of length k is exactly $1/k$. Deduce that the probability that its longest cycle has length greater than $n/2$ tends to $\ln 2$.

6. Let $C_1, \dots, C_m$ ($m \geq 3$) be convex sets in the plane, any three of which have a common point. Show that all of them have a common point.

7. Let $x_1, \dots, x_n, y_1, \dots, y_n$ be reals with every $x_i + y_j \neq 0$. Show that

$$\det\left(\frac{1}{x_i + y_j}\right)_{i,j=1}^{n} = \frac{\prod_{i<j}(x_j - x_i)(y_j - y_i)}{\prod_{i,j}(x_i + y_j)}.$$

[Please turn over]

8. Find all continuous functions $f : (0,\infty) \to (0,\infty)$ such that $f(f(x)) = 6x - f(x)$ for all $x > 0$.

9. Let $v_1, \dots, v_n$ be vectors in $\mathbb{R}^2$ with $|v_i| \le 1$. Show that signs $\varepsilon_i \in \{\pm 1\}$ can be chosen so that $|\sum \varepsilon_i v_i| \le \sqrt{2}$. For unit vectors in $\mathbb{R}^d$, show that some choice of signs gives $|\sum \varepsilon_i v_i| \le \sqrt{n}$ and some choice gives $\ge \sqrt{n}$.

10. Let $d(n)$ be the number of positive divisors of n. Show that

$$\sum_{n \le x} d(n) = x \ln x + (2\gamma - 1)x + O(\sqrt{x}),$$

where γ is Euler's constant.

ADMISSIONS QUIZ (MATHEMATICS)

MOCK Ω · OMEGA

Trinity-style mock · Paper 24 of 24 · Tier: Putnam B6 (every question)

There are ten questions below on various areas of mathematics. They are of varying levels of difficulty within this paper: some may be approachable and others hard. You are not expected to answer all of them, or to give complete answers. Attempt those that appeal to you; they will be the basis for discussion in the interview that follows. Bring what you have written with you.

1. Show that a square cannot be dissected into an odd number of triangles of equal area.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable, and suppose that for every x there is some n (depending on x) with $f^{(n)}(x) = 0$. Show that f is a polynomial.

3. Show that π^2 is irrational.

4. Let q be a prime power and let $K \subseteq \mathbb{F}_q^n$ contain a full line in every direction. Show that

$$|K| \geq \binom{q+n-1}{n}.$$

5. Show that for every integer $n \geq 1$ there is a prime p with $n < p \leq 2n$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable with $|f(x)| \leq A$ and $|f''(x)| \leq C$ for all x. Show that $|f'(x)| \leq \sqrt{2AC}$ for all x, and that the constant $\sqrt{2}$ cannot be improved.

7. Let $\lambda \geq 1$ and let $A_1, \ldots, A_m$ be distinct subsets of $\{1, \ldots, n\}$ such that $|A_i \cap A_j| = \lambda$ whenever $i \neq j$. Show that $m \leq n$.

[Please turn over]

8. Show that a regular tetrahedron cannot be cut into finitely many polyhedral pieces that can be reassembled into a cube. [Show first that $\frac{1}{\pi}\arccos\frac{1}{3}$ is irrational.]

9. Let $\|x\|$ denote the distance from x to the nearest integer. Show that

$$T(x) = \sum_{n=0}^{\infty} \frac{\|2^n x\|}{2^n}$$

is continuous on $\mathbb{R}$ but differentiable at no point.

10. Let p_d be the probability that the simple symmetric random walk on $\mathbb{Z}^d$, started at the origin, ever returns to the origin. Show that $p_1 = p_2 = 1$ and $p_d < 1$ for every $d \geq 3$.