ALPHA · OMEGA: ANSWER KEY

Final answer plus the pivotal move for each question. Attempt the paper before reading.

Mock 1 · ALPHA

1. $7! - 2 \cdot 6! = 5040 - 1440 = 3600$. Glue A, B into one block: $6!$ circular seatings, times 2 orders.
2. Substitute $x = (y - c)/m$: $my^2 - 4ay + 4ac = 0$; tangency iff discriminant 0 iff $c = a/m$. Through $(-a, k)$: $am^2 + km - a = 0$, so $m_1 m_2 = -1$.
3. $\dfrac{\pi^2}{16} - \dfrac{1}{4}$. Use $\cos^2 x = \frac{1+\cos 2x}{2}$, then parts on $x \cos 2x$.
4. 343. Inclusion-exclusion: $1000 - (500 + 200 + 142) + (100 + 71 + 28) - 14$.
5. $P(\text{DH} \mid HH) = \frac{1/3}{1/3 + \frac{2}{3}\cdot\frac{1}{4}} = \frac{2}{3}$; next head: $\frac{2}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{5}{6}$.
6. Up the plane: optimal angle to horizontal $\alpha = \frac{\pi}{4} + \frac{\beta}{2}$ (bisects the angle between the plane and the vertical).
7. Choose k men and $n - k$ women: $\sum \binom{n}{k}\binom{n}{n-k}$. Second: $n \sum \binom{n-1}{k-1}\binom{n}{n-k} = n\binom{2n-1}{n-1}$ (Vandermonde).
8. Direction vectors $(1, \frac{1}{2})$ and $(-\frac{1}{2}, 1)$ have dot product 0. $R = (\frac{4}{5}, \frac{2}{5})$, so $AR : RP = 4 : 1$.
9. Max at $(1, 1/e)$, $y \to 0^+$ as $x \to \infty$, $y \to -\infty$ as $x \to -\infty$. Solutions: $k \le 0$: one; $0 < k < 1/e$: two; $k = 1/e$: one; $k > 1/e$: none.
10. Identity by direct algebra; error squares each step (quadratic convergence). Errors roughly 0.59, 0.086, 2.5×10^{-3}, 2×10^{-6}, 1.6×10^{-12}, 10^{-24}: about 24 correct decimal places.

Mock 2 · BETA

1. Arrange M, IIII, PP: $\frac{7!}{4!2!} = 105$; choose 4 of the 8 gaps for S: $\binom{8}{4} = 70$. Total 7350.
2. $t = \sqrt{x - 1}$: $|t - 2| + |t - 3| = 1 \iff 2 \le t \le 3 \iff 5 \le x \le 10$.
3. $\pi/4$. Add the integral to itself under $x \mapsto \frac{\pi}{2} - x$: integrand sums to 1.
4. $n^5 - n = (n-1)n(n+1)(n^2+1)$; divisible by 2, 3, and by 5 (Fermat). n odd: $(n-1)(n+1)$ gives 8, $n^2 + 1$ gives 2: total $16 \cdot 3 \cdot 5 = 240$.
5. Fix the first point; the triangle misses the centre iff all three lie in a semicircle, probability $\frac{3}{4}$. Acute iff it contains the centre: also $\frac{1}{4}$.
6. Moments about the foot: $R\,l \sin\theta = \frac{W}{2} l \cos\theta$, friction $F = R \le \mu W$. Mass cancels: every force scales with W.
7. Induction, or $\det \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = (-1)^n$. A common factor d of F_n, F_{n+1} divides $(-1)^n$.
8. $\frac{1}{3}$. The region is the hexagon cut out by the six perpendicular bisectors: apothem $\frac{1}{2}$ against $\frac{\sqrt{3}}{2}$, area ratio $(\frac{1/2}{\sqrt{3}/2})^2 = \frac{1}{3}$.
9. $n \ln(1 + \frac{1}{n}) = 1 - \frac{1}{2n} + O(n^{-2})$; exponentiate: $e(1 - \frac{1}{2n})$.
10. $\max x^{1/x} = e^{1/e}$ at $x = e$. So $\pi^{1/\pi} < e^{1/e}$, raise to $e\pi$: $\pi^e < e^\pi$.

Mock 3 · GAMMA

1. Bad paths biject (reflect after first touch of $y = x + 1$) with paths to $(n - 1, n + 1)$: $\binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$.
2. $\tan 3\theta = \sqrt{3}$ for $\theta = 20°, 80°, 140°$, so their tangents are the roots of $t^3 - 3\sqrt{3}t^2 - 3t + \sqrt{3} = 0$. Product of roots: $\tan 20° \tan 80° (-\tan 40°) = -\sqrt{3}$.

3. Parts. $e^x \le e$ gives the bound. Then $I_n = (-1)^n n!\big(e\sum_{k\le n}\frac{(-1)^k}{k!} - 1\big)$, and $I_n/n! \to 0$.
4. None: $2026 \equiv 2 \pmod 4$ but $x^2 - y^2 \equiv 0, 1, 3 \pmod 4$. Representable iff $n \not\equiv 2 \pmod 4$.
5. $P(\text{even}) = \frac{5/36}{1-25/36} = \frac{5}{11}$. Two consecutive sixes: $E = 6 + 36 = 42$.
6. Energy $v^2 = 2ga(1-\cos\theta)$; leaves when $mg\cos\theta = mv^2/a$: $\cos\theta = \frac{2}{3}$, $v = \sqrt{2ga/3}$.
7. $(2^{100} - 1)x + 2 - 2^{100}$ (evaluate at $x = 1, 2$).
8. Area 84 (Heron, $s = 21$), $r = 84/21 = 4$, $R = \frac{abc}{4\Delta} = \frac{65}{8}$.
9. $y = 1 + \frac{3}{x^2-4}$; asymptotes $x = \pm 2$, $y = 1$; max $(0, \frac{1}{4})$. Range $(-\infty, \frac{1}{4}] \cup (1, \infty)$.
10. Telescope $\frac{1}{n} - \frac{1}{n+1}$ and $\frac{1}{2}(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)})$. Then $\sum \frac{1}{n^2} < 1 + \sum_{n\ge 1}\frac{1}{n(n+1)} = 2$.

Mock 4 · DELTA

1. Pairings of $2m$ people: $(2m-1)!!$. Inclusion-exclusion over couples forced together: $\sum_k (-1)^k \binom{5}{k}(9-2k)!! = 945 - 525 + 150 - 30 + 5 - 1 = 544$.
2. $\mathrm{Re}(1 + e^{i\theta})^n$ and $1 + e^{i\theta} = 2\cos\frac{\theta}{2}\, e^{i\theta/2}$.
3. $x \mapsto 1/x$ swaps them. Sum $= \int \frac{1+x^{-2}}{(x-x^{-1})^2+2}dx = \int_{-\infty}^{\infty}\frac{du}{u^2+2} = \frac{\pi}{\sqrt{2}}$; each is $\frac{\pi}{2\sqrt{2}}$.
4. $7^4 = 2401 \equiv 1 \pmod{100}$, so $7^{2026} \equiv 49$. $3^{20} \equiv 1 \pmod{100}$, $3^{27} \equiv 3^7 = 2187$: 87.
5. $E[\mathrm{HH}] = 6$, $E[\mathrm{HT}] = 4$. HH overlaps itself: a failed attempt (HT) wastes the progress, whereas for HT, once you have H any T completes it and extra H's keep you ready.
6. Energy, or $L\ddot{x} = g(2x - L)$. Check: lost PE $= \rho g\, a(L - a)$.
7. $\int_0^1 (1-x)^n dx = \frac{1}{n+1}$; algebraically $\frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}\binom{n+1}{k+1}$.
8. Segment area $\theta - \frac{1}{2}\sin 2\theta = \frac{\pi}{4}$. Numerically $\theta \approx 1.15$.
9. $f(x) = x\ln x$ increasing on $(1, \infty)$ from 0 to ∞. $w\ln w = L := \ln N$ gives $w \approx \frac{L}{\ln L - \ln\ln L}$.
10. $u = x + y$: $u' = 1 + u^2$, $y = \tan x - x$, on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Mock 5 · EPSILON

1. $a_n = a_{n-1} + a_{n-2}$ (contains n or not): $a_n = F_{n+2}$; $a_{10} = 144$.
2. Apollonius: $m_a^2 = \frac{2b^2+2c^2-a^2}{4}$; sum.
3. $\frac{\pi}{8}\ln 2$. After the substitutions, $\ln(1 + \tan(\frac{\pi}{4} - \theta)) = \ln 2 - \ln(1 + \tan\theta)$.
4. If $p_1, \ldots, p_k$ are all such primes, $4p_1 \cdots p_k - 1$ has a prime factor $\equiv 3 \pmod 4$ not in the list.
5. Triangle iff every piece $< \frac{1}{2}$; the region is $\frac{1}{4}$ of the square. Second model: $\int$ over the first break gives $2\ln 2 - 1 \approx 0.386$.
6. $\dot{r}$ constant; $\frac{d}{dt}(r^3 v) = r^3 g$ gives $v = \frac{g}{4}t$: acceleration $\frac{g}{4}$. Swept version: $\dot{r} \propto v$, try $v = at$: $a = \frac{g}{7}$.
7. Newton: $p_1 = 0, p_2 = 2, p_3 = 3$, $p_k = p_{k-2} + p_{k-3}$: $p_5 = 5$. With $y = \frac{1}{1-x}$: $y^3 + 2y^2 - 3y + 1 = 0$, sum -2.
8. $r = \frac{3V}{S} = \frac{1/2}{3/2+\sqrt{3}/2} = \frac{3-\sqrt{3}}{6}$.
9. Three: $x = 2$, $x = 4$, and one in $(-1, 0)$ (≈ -0.767).
10. $1 + (a + b + ab) = (1 + a)(1 + b)$: $\prod(1 + x)$ is invariant. Final number $21! - 1$.

Mock 6 · ZETA

1. Transfer/first-block argument gives $a_n = 3a_{n-1} + 2(a_{n-2} + \cdots + a_0)$, equivalently $a_n = 4a_{n-1} - a_{n-2}$. $a_2 = 11, a_3 = 41, a_4 = 153$.

2. $1 + \frac{1}{a} = \frac{a+1}{a} = \frac{2a+b+c}{a} \geq \frac{4\sqrt[4]{a^2bc}}{a}$ (AM-GM); multiply the three.

3. $\frac{\pi^2}{4}$. $x \mapsto \pi - x$ gives $\frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx = \frac{\pi}{2} \cdot \frac{\pi}{2}$.

4. Pair each residue with its inverse; only ± 1 are self-inverse. $100! \equiv -1$ and $98 \cdot 99 \cdot 100 \equiv -6$, so $97! \equiv 6^{-1} \equiv 17 \pmod{101}$.

5. Any H preceded by T means THH arrives first unless the first three tosses are HHH. $E[\mathrm{HHH}] = 14$, $E[\mathrm{HTH}] = 10$ (sum of 2^k over self-overlaps).

6. Contact requires platform acceleration downward $\leq g$: $A\omega^2 \leq g$. Otherwise leaves on the way up at height g/ω^2 above the centre.

7. $a^k - b^k = (a-b)(\dots)$. $2 \nmid 3$. Second: $P - 1 = (x-1)(x-2)(x-3)Q$ gives $6Q(4) = 1$: impossible.

8. Routh's theorem with ratio 2: $\frac{(2-1)^2}{2^2+2+1} = \frac{1}{7}$ (or mass points / coordinates).

9. Compare $\frac{1}{k}$ with $\int \frac{dx}{x}$. $H_n - \ln n$ decreases since $\frac{1}{n+1} < \ln\frac{n+1}{n}$, bounded below by $\ln\frac{n+1}{n} > 0$. $H_{2n} - H_n$ is a Riemann sum for $\int_1^2 \frac{dx}{x}$.

10. $(xy)'' + xy = 0$, i.e. $u'' + u = 0$: $y = \frac{A\sin x + B\cos x}{x}$.

Mock 7 · ETA

1. Where does n go? To i; then either $i \mapsto n$ (D_{n-2}) or not (D_{n-1}). Inclusion-exclusion gives the sum; proportion $\to 1/e$.

2. $\frac{z^{2n}-1}{z^2-1} = \prod_{k=1}^{n-1}(z - \omega^k)(z - \omega^{-k})$, $\omega = e^{i\pi/n}$; set $z = 1$ and use $|1 - e^{i\phi}| = 2\sin\frac{\phi}{2}$.

3. Divide: $\frac{x^4(1-x)^4}{1+x^2} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}$. Integrand > 0, so $\pi < \frac{22}{7}$.

4. Squares are $0, 1 \pmod 3$, so $3 \mid x, y$, then $3 \mid z$: infinite descent.

5. $P_k = pP_{k+1} + qP_{k-1}$, $P_0 = 0, P_N = 1$. Fair case k/N. As $N \to \infty$: 0 if $p \leq \frac{1}{2}$, $1 - (q/p)^k$ if $p > \frac{1}{2}$.

6. No horizontal force. With $y = a\cos\theta$: $mga = \frac{1}{2}m\dot{y}^2 + \frac{1}{2} \cdot \frac{ma^2}{3}\dot{\theta}^2$ and at $\theta = \frac{\pi}{2}$, $\dot{y} = -a\dot{\theta}$: $v = \sqrt{3ga/2}$.

7. $g(x) = f(x) - \frac{x^2}{2}$ is additive and continuous: $f(x) = \frac{x^2}{2} + cx$.

8. Centres form an equilateral triangle of side 2, circumradius $\frac{2}{\sqrt{3}}$: $r = \frac{2}{\sqrt{3}} - 1$.

9. True. Need $\{n\log_{10} 2\} \in [\log_{10} 2.026, \log_{10} 2.027)$; $\log_{10} 2$ irrational so $\{n\log_{10} 2\}$ is dense.

10. $\frac{1}{n-\frac{1}{2}} - \frac{1}{n+\frac{1}{2}} = \frac{1}{n^2-\frac{1}{4}}$. Sum telescopes to $1 + \frac{2}{3}$.

Mock 8 · THETA

1. Generating functions $\prod(1 + x^k) = \prod\frac{1-x^{2k}}{1-x^k} = \prod\frac{1}{1-x^{2k-1}}$. Or Glaisher: merge equal parts in pairs. $n = 8$: six each (8, 71, 62, 53, 521, 431; 71, 53, 5111, 3311, 311111, 1^8).

2. $z = e^{2\pi ik/5}$. $c = 2\cos\frac{2\pi}{5}$ satisfies $c^2 + c - 1 = 0$ from $1 + z + z^2 + z^3 + z^4 = 0$ divided by z^2.

3. $I'(a) = -\int_0^\infty e^{-ax}\sin x\,dx = -\frac{1}{1+a^2}$; $I(\infty) = 0$ gives $I(a) = \frac{\pi}{2} - \arctan a$; need continuity of I at $a = 0$.

4. Let p be the smallest prime factor of n. $\mathrm{ord}_p 2$ divides n and $p - 1$, so it is 1: $p \mid 1$, contradiction.

5. (i) $\frac{1}{3}$, (ii) $\frac{1}{2}$, (iii) $\frac{1}{4}$. "At random" is not a well-defined distribution until a model is chosen.

6. Rotating frame: $\sin\theta(a\omega^2\cos\theta - g) = 0$. $\theta = 0$ (bottom), π (top), $\cos\theta = \frac{g}{a\omega^2}$. When the side equilibria exist they are stable and the bottom is unstable; otherwise the bottom is stable.

7. $\sum r_i^2 = a_{n-1}^2 - 2a_{n-2} = 0$ with real r_i.

8. Split into three triangles from the point: $\frac{1}{2}s(d_1+d_2+d_3) =$ area. Tetrahedron: sum of distances to faces constant. Square: $PA + PC \geq AC$, $PB + PD \geq BD$, equality at centre: yes.

9. $g(x) = x^n + x - 1$ increasing, $g(0) < 0 < g(1)$. $\varepsilon = 1 - x_n = (1-\varepsilon)^n \approx e^{-n\varepsilon}$, so $n\varepsilon e^{n\varepsilon} \approx n$: $n\varepsilon \approx \ln n - \ln\ln n$.

10. Binet: $\sum \binom{n}{k}\varphi^k = (1+\varphi)^n = \varphi^{2n}$, same for $\bar{\varphi}$.

Mock 9 · IOTA

1. Write each number as $2^a b$ with b odd; only n odd parts exist. Counterexample: $\{n+1, \ldots, 2n\}$.

2. $x = 2\cos\theta$: $2\cos 3\theta = 2\cos\frac{\theta}{2}$. Solutions $x = 2,\ 2\cos\frac{4\pi}{5},\ 2\cos\frac{4\pi}{7}$.

3. $x^{-x} = e^{-x\ln x} = \sum \frac{(-x\ln x)^k}{k!}$; $\int_0^1 x^k(\ln x)^k dx = \frac{(-1)^k k!}{(k+1)^{k+1}}$.

4. $n(n+1)(n+2)(n+3) = (n^2+3n+1)^2 - 1$ lies strictly between consecutive squares.

5. Indicators: $E[X] = n \cdot \frac{1}{n}$. $E[X(X-1)] = n(n-1) \cdot \frac{1}{n(n-1)} = 1$, so Var $= 1 + 1 - 1 = 1$.

6. $y = x\tan\alpha - \frac{gx^2}{2u^2}(1+\tan^2\alpha)$ is a quadratic in $\tan\alpha$; real root iff discriminant ≥ 0, which is the parabola condition.

7. $f(x^2) = f(x)^2 \geq 0$ makes f monotone; additive and monotone gives $f(x) = cx$, $c \in \{0, 1\}$. Answer $f \equiv 0$ or $f(x) = x$.

8. Area $= \frac{\sqrt{3}}{4}s^2$ with $s^2 \in \mathbb{Z}$ is irrational, but Pick/shoelace makes it rational. In 3D: $(1,0,0), (0,1,0), (0,0,1)$.

9. $c = \frac{1}{2}$: compare coefficients $\frac{1}{(2n)!} \leq \frac{1}{2^n n!}$; near 0, $\cosh x = 1 + \frac{x^2}{2} + \ldots$ forces $c \geq \frac{1}{2}$.

10. $a_{n+1}^2 = a_n^2 + 2 + a_n^{-2}$. Lower bound gives $a_{1000}^2 \geq 1999 > 1936$. Upper: $a_{1000}^2 \leq 1999 + \sum_{k<1000} \frac{1}{2k-1} < 1999 + 5 < 2025$.

Mock 10 · KAPPA

1. Each element lies in half the subsets. $\sum |A \cap B| = n4^{n-1}$, $\sum |A \cup B| = 3n4^{n-1}$.

2. Compare derivatives. $\sum \ln(1 + \frac{k}{n^2}) = \frac{n+1}{2n} + O(\frac{1}{n})$: limit $\sqrt{e}$.

3. $I = J$ by $x \mapsto \frac{\pi}{2} - x$. $2I = \int_0^{\pi/2} \ln\sin 2x\, dx - \frac{\pi}{2}\ln 2$, and $\int_0^{\pi/2} \ln\sin 2x\, dx = \frac{1}{2}\int_0^{\pi} \ln\sin u\, du = I$. So $I = -\frac{\pi}{2}\ln 2$.

4. $F_n - 2 = 2^{2^n} - 1 = (2^{2^{n-1}} - 1)(2^{2^{n-1}} + 1)$, induct. A common factor divides 2; F_m odd.

5. $P = \frac{1}{\pi}\int_0^{\pi} \frac{l\sin\theta}{d} d\theta = \frac{2l}{\pi d}$. Drop many needles, $\pi \approx \frac{2lN}{d\,\#\text{crossings}}$.

6. $M = 3m$. With B fixed the lower system accelerates, so B's string tension is $\frac{4m_1m_2}{m_1+m_2}g = 3mg < 4mg$.

7. $\sum kx^k = \frac{x}{(1-x)^2}$, differentiate. 6 and 26.

8. Bretschneider: $K^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\frac{A+C}{2}$. Max $\sqrt{4\cdot 3\cdot 2\cdot 1} = 2\sqrt{6}$, cyclic.

9. (i) True: Zeckendorf, greedy algorithm. (ii) True: balanced ternary, induct on $|n|$ via n mod 3.

10. Variation of parameters: $y = A\cos x + B\sin x + x\sin x + \cos x\ln\cos x$.

Mock 11 · LAMBDA

1. Last part 1 or 2: $c_n = c_{n-1} + c_{n-2}$. Odd parts: last part 1 (gives o_{n-1}) or add 2 to it (gives o_{n-2}).

2. Expand. n even: even. $n = 2k+1$: $n^4 + 4\cdot(2^k)^4$ with both factors > 1.

3. $I'(t) = \int_0^1 x^t dx = \frac{1}{t+1}$, $I(0) = 0$: $I(1) = \ln 2$.

4. $(2+\sqrt{3})^n + (2-\sqrt{3})^n$ is an even integer; $0 < (2-\sqrt{3})^n < 1$.

5. $P(A) = \frac{1/6}{1-\frac{5}{6}\cdot\frac{4}{6}} = \frac{3}{8}$. B is favoured, $\frac{5}{8}$.

6. $a = \frac{g\sin\alpha}{1+I/(mr^2)}$: disc $\frac{2}{3}g\sin\alpha$, hoop $\frac{1}{2}g\sin\alpha$. Friction $\frac{1}{3}mg\sin\alpha$: $\mu \geq \frac{1}{3}\tan\alpha$.

7. $P \equiv 0$ or $P = (x^2-x)^k$, $k \geq 0$. Roots r force r^2 and $(r-1)^2$ to be roots; only $0, 1$ survive with a finite root set.

8. $\ell = \frac{2bc\cos(A/2)}{b+c} = \frac{2\cdot 15\cdot\frac{1}{2}}{8} = \frac{15}{8}$.

9. 2^{100}. $3^{63} = 2187^9$, $2^{100} = 2\cdot 2048^9$, and $(1+\frac{139}{2048})^9 < e^{9\cdot 139/2048} = e^{0.61} < 2$.

10. $x_0 = 2 + \frac{1}{2}$; $(t+t^{-1})^2 - 2 = t^2 + t^{-2}$: $x_n = 2^{2^n} + 2^{-2^n}$.

Mock 12 · MU

1. Person P knows or does not know ≥ 3 others; examine those three. Five: pentagon (friends) and pentagram (strangers).

2. Take logs: $\sum a \ln a \geq \frac{\sum a}{3}\sum \ln a$, which is Chebyshev (sequences a and $\ln a$ similarly ordered).

3. $x = a\,t$: $\frac{1}{a}\int \frac{\ln a + \ln t}{1+t^2}dt = \frac{\pi \ln a}{2a}$ since $\int_0^\infty \frac{\ln t}{1+t^2} = 0$ $(t \mapsto 1/t)$.

4. Let 2^k be the largest power of 2 up to n. With $L = \operatorname{lcm}(1,\ldots,n)$, every L/j is even except $L/2^k$, so LH_n is odd while L is even.

5. $u_{2n} = \binom{2n}{n}4^{-n}$; the bound by induction on the ratio $\frac{u_{2n+2}}{u_{2n}} = \frac{2n+1}{2n+2}$. $\sum u_{2n} = \infty$ is the expected number of visits.

6. Energy with extension e: $mg(l+e) = \frac{2mg}{2l}e^2$, $e = \frac{1+\sqrt{5}}{2}l$. Greatest depth $\frac{3+\sqrt{5}}{2}l$.

7. Real roots have even multiplicity; pair complex roots: $P = c\,Q(x)\overline{Q(x)}$, write $Q = A + iB$.

8. Affine map from the circle: max triangle in a circle is equilateral, area $\frac{3\sqrt{3}}{4}$; scale by ab: $\frac{3\sqrt{3}}{4}ab$.

9. True: among $1, 11, 111, \ldots$ $(n+1$ numbers) two agree mod n; subtract.

10. $-\ln(1-x) = \sum \frac{x^n}{n}$: $\ln 2$. $\sum H_n x^n = \frac{-\ln(1-x)}{1-x}$: $2\ln 2$.

Mock 13 · NU

1. Opposite corners share a colour; each domino covers one of each. Gomory: the board is one Hamiltonian cycle of alternating colours; removing opposite colours leaves two even-length paths.

2. $g(x) = f(x+\frac{1}{n}) - f(x)$ sums to 0 over $x = 0, \frac{1}{n}, \ldots$, so changes sign. Counterexample for $c \neq \frac{1}{n}$: $f(x) = \sin^2\frac{\pi x}{c} - x\sin^2\frac{\pi}{c}$.

3. 0 if $|a| < 1$; $2\pi\ln|a|$ if $|a| > 1$. Factor $|1-ae^{ix}|^2$ and expand $\ln|1-ae^{ix}| = -\operatorname{Re}\sum\frac{a^k e^{ikx}}{k}$; for $|a| > 1$ pull out a^2.

4. CRT with $2n$ distinct primes: $x + i \equiv 0 \pmod{p_i q_i}$ for $i = 1, \ldots, n$.

5. Event "all within the semicircle clockwise from point i" has probability 2^{1-n}; these n events are disjoint.

6. Let D be the distance and $X = D\cos\phi$ the gap along the rabbit's direction. $\dot{D} = v\cos\phi - w$, $\dot{X} = v - w\cos\phi$, so $wD + vX$ falls at rate $w^2 - v^2$ from wd.

7. Trace of $AB - BA$ is $0 \neq n$. Idempotent: diagonalisable with eigenvalues $0, 1$.

8. Convex hull with ≥ 4 vertices: done. Triangle with two interior points: the line through them leaves two vertices on one side; those four points are in convex position.

9. $\min(i,j) = \sum_k [k \le i][k \le j]$: the form is $\sum_k \left(\sum_{i \ge k} x_i\right)^2$.

10. $x_n \downarrow 0$. $\frac{1}{x_{n+1}^2} - \frac{1}{x_n^2} \to \frac{1}{3}$ (from $\sin x = x - \frac{x^3}{6} + \dots$), then Cesaro.

Mock 14 · XI

1. Choose the set S of values in the first run: any nonempty proper subset except initial segments $\{1..k\}$, giving $2^n - 2 - (n-1)$.

2. $\int f \cdot p = 0$ for all polynomials p; take $p_k \to f$ uniformly: $\int f^2 = 0$.

3. $\ln(1-x) = -\sum \frac{x^n}{n}$, $\int_0^1 x^n \ln x = -\frac{1}{(n+1)^2}$: $\sum \frac{1}{n(n+1)^2} = 2 - \frac{\pi^2}{6}$.

4. $\frac{n}{\varphi(n)} = \prod \frac{p}{p-1}$ must be an integer: $n = 1, 2^a, 2^a 3^b$ $(a, b \ge 1)$.

5. Any specific sequence with r reds among n draws has probability $\frac{r!(n-r)!}{(n+1)!}$; times $\binom{n}{r}$ gives $\frac{1}{n+1}$.

6. Depth $\propto t$ (hours since start t_0 before noon); $x = k \ln \frac{t}{t_0}$. $\frac{t_0+1}{t_0} = \left(\frac{t_0+2}{t_0+1}\right)^2$ gives $t_0^2 + t_0 = 1$: $t_0 = \frac{\sqrt{5}-1}{2}$ h, about 11:22:55.

7. Eigenvalues satisfy $\lambda^3 - \lambda - 1 = 0$: one real root > 0, complex ones in conjugate pairs; determinant is a product $r^a |z|^{2b} > 0$.

8. Rotate $\triangle APB$ by 60° about B (Torricelli). Minimum $d^2 = \frac{a^2+b^2+c^2}{2} + 2\sqrt{3}\Delta = 25 + 12\sqrt{3}$.

9. Nesbitt by Cauchy-Schwarz. Second sum: each term in $(0,1)$ and the sum plus its cyclic reverse equals 3; $a = 1, b = t, c = t^2$ with $t \to 0, \infty$ approaches 1 and 2.

10. $\cot^2 \frac{k\pi}{2m+1}$ are the roots of the polynomial from $\operatorname{Im}(\cot\theta + i)^{2m+1} = 0$. Sandwich $\sum \frac{1}{k^2}$ between $\frac{\pi^2}{(2m+1)^2}\cdot$ the two sums.

Mock 15 · OMICRON

1. Label each term with (longest increasing ending here, longest decreasing ending here); labels are distinct pairs. Counterexample: n decreasing blocks of n increasing terms.

2. f injective, so monotone; then $f \circ f$ increasing but $-x$ decreasing. A discontinuous one exists: pair up $(a, b, -a, -b)$ 4-cycles.

3. Parts: $[x \ln \sin x] - \int \ln \sin x = \frac{\pi}{2} \ln 2$.

4. Fermat's descent via Pythagorean triples: from a minimal (x, y, z) build a smaller one.

5. Gaps are exchangeable; shortest $= \frac{1}{n} \cdot \frac{1}{n}$ (min of spacings: $E = \frac{1}{n^2}$); longest $= \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k}$.

6. Horizontal tension constant T_0; $\frac{dy}{dx} = \frac{\rho g s}{T_0}$, differentiate: $y'' = \frac{1}{c}\sqrt{1 + y'^2}$.

7. $x^T H x = \int_0^1 \left(\sum x_i t^{i-1}\right)^2 dt > 0$.

8. Take a longest chord AB; the polygon lies between perpendiculars at A, B. Farthest points C, D on each side give a rectangle of area $2(\triangle ABC + \triangle ABD) \le 2$.

9. AM-GM / Bernoulli on ratios, or $f(x) = (x+1)\ln(1 + \frac{1}{x})$ decreasing.

10. Yes (Dirichlet test: partial sums of $\sin n$ bounded). No: $|\sin n| \ge \sin^2 n = \frac{1 - \cos 2n}{2}$ and $\sum \frac{\cos 2n}{n}$ converges.

Mock 16 · PI

1. Each maximal chain meets $\mathcal{F}$ at most once; A lies on $|A|!(n - |A|)!$ of the $n!$ chains (LYM).

2. Pick c with $f(c) = 1 - c$ (IVT). MVT on $[0, c]$ and $[c, 1]$: $f'(a) = \frac{1-c}{c}$, $f'(b) = \frac{c}{1-c}$.

3. Mass concentrates at 1. Limit $\frac{1}{2}$; $\int_0^1 \frac{x^n}{1+x} dx = \frac{1}{2n} - \frac{1}{4n^2} + O(n^{-3})$ (parts twice).

4. Pair $\frac{1}{k}+\frac{1}{p-k}=\frac{p}{k(p-k)}$; then $\sum\frac{1}{k(p-k)}\equiv-\sum\frac{1}{k^2}\equiv 0 \pmod p$ since $\sum k^2\equiv 0$ for $p>3$.

5. $P(r)=\frac{r}{n}\sum_{k=r+1}^{n}\frac{1}{k-1}\approx -x\ln x$ with $x=\frac{r}{n}$; maximised at $x=\frac{1}{e}$.

6. Bounce speeds $e^k u$ with $u=\sqrt{2gh}$. Time $\sqrt{\frac{2h}{g}}\cdot\frac{1+e}{1-e}$; distance $h\frac{1+e^2}{1-e^2}$.

7. $G=D\Phi D^T$ with $D_{id}=[d\mid i]$ lower unitriangular, $\Phi=\operatorname{diag}\varphi(d)$.

8. Power of I with respect to circumcircle: $R^2-OI^2=IA\cdot IM=\frac{r}{\sin(A/2)}\cdot 2R\sin\frac{A}{2}=2Rr$.

9. $(a_1\cdots a_n)^{1/n}\le\frac{\sum_{k\le n}ka_k}{n(n!)^{1/n}}\le e\frac{\sum_{k\le n}ka_k}{n(n+1)}$, swap sums: $\sum_{n\ge k}\frac{1}{n(n+1)}=\frac{1}{k}$.

10. $\frac{4}{3}+\frac{2\pi\sqrt{3}}{27}$. Use $\binom{2n}{n}^{-1}=(2n+1)\int_0^1 t^n(1-t)^n dt$ and sum a geometric series.

Mock 17 · RHO

1. Prüfer code: a bijection between trees and sequences in $[n]^{n-2}$.

2. Let f take its max at $a<b$ (twice), analyse values just below the max: three preimages appear. Three times: yes, a zigzag.

3. Expand $\frac{1}{1-xy}=\sum(xy)^n$. $\frac{x^3}{e^x-1}=\sum x^3e^{-nx}$, $\int=\frac{6}{n^4}$, giving $\frac{\pi^4}{15}$.

4. $a=(\frac{p-1}{2})!$ (Wilson). Pigeonhole: two pairs agree mod p; difference (u,v) gives $u^2+v^2\equiv 0$, $0<u^2+v^2<2p$.

5. Build the cycle of 1: at each step close with probability $\frac{1}{\text{remaining}}$. Cycles: $\sum\frac{1}{k}$ by the same process.

6. Energy gives $\dot\theta^2=\frac{2g}{l}(\cos\theta-\cos\alpha)$; substitute $\sin\frac{\theta}{2}=k\sin\phi$. Increasing in k; $K\to\infty$ as $k\to 1$. $T\approx 2\pi\sqrt{l/g}\,(1+\frac{\alpha^2}{16})$.

7. Block matrices $\begin{pmatrix}I & -A\\ 0 & I\end{pmatrix}$, or perturb A to be invertible and use $BA=A^{-1}(AB)A$ plus continuity.

8. Take a point P and line ℓ at minimal positive distance; if ℓ had 3 points, two lie on one side of the foot of the perpendicular, giving a smaller distance.

9. Convex in each variable, so max at $x_i\in\{0,1\}$: $k(n-k)$. Equality: $\lfloor n/2\rfloor$ zeros and the rest ones.

10. Trapezium error bounds show a_n decreasing and bounded. Wallis: $\frac{(2^n n!)^4}{((2n)!)^2(2n+1)}\to\frac{\pi}{2}$ gives $e^{2c}=2\pi$.

Mock 18 · SIGMA

1. A vertex has 16 edges, ≥ 6 of one colour; among those 6 neighbours use $R(3,3)=6$ or find that colour.

2. One sign change at c forces $\int(x-c)f\neq 0$. General: n vanishing moments give $\ge n$ sign changes.

3. Parts: $\int\frac{\sin^2 x}{x^2}=\int\frac{\sin 2x}{x}$. For the cube use $\sin^3 x=\frac{3\sin x-\sin 3x}{4}$ and parts twice.

4. Gaussian integers / $(a^2+b^2)(c^2+d^2)$ identity for sufficiency. Necessity: $q\mid a^2+b^2$ with $q\equiv 3$ forces $q\mid a,b$ (since -1 is a non-residue).

5. Reflection principle: bad sequences starting with B biject with bad ones starting with A. $P=1-\frac{2b}{a+b}$.

6. $\mathbf{r}\times\ddot{\mathbf{r}}=0$. Binet's equation; $F=\frac{k}{r^2}$ gives $u=\frac{k}{mh^2}+A\cos(\theta-\theta_0)$.

7. Polynomial in a_n vanishing at $a_1,\dots,a_{n-1}$. For independence differentiate $\sum c_i e^{a_i x}=0$ repeatedly at 0.

8. Inversion centred at A turns it into the triangle inequality $B'C'+C'D'\ge B'D'$.

9. With $B_n=A_n/n$: $B_n^2-2a_nB_n\le(n-1)B_{n-1}^2-nB_n^2$; summing, $\sum B_n^2\le 2\sum a_nB_n\le 2\sqrt{\sum a_n^2}\sqrt{\sum B_n^2}$.

10. $b_n = e^{a_n}$: $b_{n+1} = b_n e^{1/b_n} = b_n + 1 + O(\frac{1}{b_n})$, so $b_n = n + O(\ln n)$ and $a_n - \ln n \to 0$.

Mock 19 · TAU

1. Hall's condition: a set S of left vertices sends $k|S|$ edges into $N(S)$, which absorbs at most $k|N(S)|$. Cards: piles versus ranks, 4-regular multigraph.

2. $(e^x f)' = e^x(f + f')$ and L'Hôpital (or direct ε argument). $f + 2f'$: yes, use $e^{x/2}$. $f - f'$: no, $f = e^x$.

3. Expand geometrically. The map has Jacobian $1 - x^2y^2$ and sends the square onto the triangle $u, v > 0$, $u + v < \frac{\pi}{2}$, area $\frac{\pi^2}{8}$. Then $\zeta(2) = \frac{4}{3} \cdot \frac{\pi^2}{8}$.

4. Induct: given N for n, one of $d \cdot 10^n + N$ ($d \in \{1,3,5,7,9\}$) is divisible by 5^{n+1}, since $d \cdot 2^n$ runs over all residues mod 5.

5. $E_k = 1 + \frac{E_{k-1}+E_{k+1}}{2}$, $E_{\pm n} = 0$: $E_k = n^2 - k^2$. Reaching $+1$: $P(\text{hit } +1 \text{ before } -n) = \frac{n}{n+1} \to 1$. Expected time $\geq$ expected time to hit $\{-n, +1\}$, which is n, for every n.

6. Circumscribed rectangle in direction θ has sides $w(\theta), w(\theta + \frac{\pi}{2})$; the difference changes sign between θ and $\theta + \frac{\pi}{2}$. IVT.

7. Eigenvalues 0 and $\pm ib_j$: $\det(I + A) = \prod(1 + b_j^2) \geq 1$.

8. Exterior angles sum to $360°$ and each acute angle has exterior angle $> 90°$. Dart: angles $30°, 30°, 30°, 270°$. For simple n-gons the maximum is $\lfloor \frac{2n}{3} \rfloor + 1$.

9. $x \mapsto \frac{x}{\sqrt{1-x}}$ is convex; Jensen at $x = \frac{1}{n}$.

10. $2\ln 2 - 1$. The summand is 1 iff $\{\frac{n}{k}\} \geq \frac{1}{2}$; Riemann sum for $\int_0^1 (\lfloor \frac{2}{x} \rfloor - 2\lfloor \frac{1}{x} \rfloor) dx$.

Mock 20 · UPSILON

1. Chevalley-Warning: two forms of degree $p - 1$ in $2p - 1 > 2(p - 1)$ variables, so the number of solutions is $\equiv 0 \pmod{p}$; a nonzero solution has exactly p nonzero coordinates, and their a_i sum to 0.

2. If $a_n \geq 2^{-n}$: $a_n^{1-1/n} = a_n / a_n^{1/n} \leq 2a_n$. Otherwise $a_n^{1-1/n} < 2^{-(n-1)}$.

3. $(\arcsin x)^2 = \frac{1}{2} \sum_{n \geq 1} \frac{(2x)^{2n}}{n^2 \binom{2n}{n}}$; set $x = \frac{1}{2}$: $\frac{\pi^2}{36} = \frac{1}{2}S$.

4. Group $1, \ldots, n$ by $\gcd(k, n)$. Elements of order d number 0 or $\varphi(d)$; the counts sum to $p - 1 = \sum \varphi(d)$, so all are $\varphi(d)$, including $d = p - 1$.

5. Known value $\frac{11}{144} \approx 0.0764$; compute via fixing the leftmost point, or Crofton's method. (Monte Carlo agrees to 4 d.p.)

6. Fourier: $\sum n^2 |c_n|^2 \geq \sum |c_n|^2$ when $c_0 = 0$. Arc-length parametrise, $L^2 - 4\pi A = \cdots \geq 0$ via Wirtinger on x, y.

7. Mod 2, $A \equiv J - I$, $\det(J - I) = (-1)^{n-1}(n - 1)$ is odd for even n.

8. Each tile has $\iint e^{2\pi i(x+y)} = 0$ (one factor $\int_a^{a+k} e^{2\pi i x} dx = 0$). Sum is 0 for the big rectangle, so it has an integer side.

9. $a^2 + \sqrt{a} + \sqrt{a} \geq 3a$ (AM-GM). Sum: $\sum a^2 + 2\sum \sqrt{a} \geq 9 = (\sum a)^2 = \sum a^2 + 2\sum ab$.

10. $\frac{1}{e}$ (Riemann sum $\frac{1}{n} \sum \ln \frac{k}{n} \to -1$). $\frac{4}{e}$ by the same method or Stirling.

Mock 21 · PHI

1. Graham-Pollak: $\sum_{i<j} x_i x_j = \sum_k L_k(x) R_k(x)$. If fewer than $n - 1$ pieces, the system $\sum x_i = 0$, $L_k = 0$ has a nonzero solution, but then $0 = \frac{1}{2}[(\sum x_i)^2 - \sum x_i^2] < 0$.

2. If $x_{n+1} - x_n \to 0$ but no convergence, limit set is an interval $[a, b]$ of fixed points; the orbit enters it and cannot leave, contradiction.

3. Split at 1, expand $\frac{1}{1+x}$: $\sum(-1)^n\left(\frac{1}{s+n}+\frac{1}{n+1-s}\right)=\frac{\pi}{\sin \pi s}$ (partial fractions of cosecant). Or Beta function: $B(s,1-s)$.
4. $p \mid \Phi_n(a)$, $p \nmid n$: order of a mod p is exactly n, so $n \mid p-1$. Apply to $a=kN$ for large products N.
5. Count coprime pairs via Möbius: $P_N \to \sum \frac{\mu(d)}{d^2}=\frac{1}{\zeta(2)}$.
6. Consecutive edge vectors u, v are integer with $|u|=|v|$ and $u \cdot v=|u|^2 \cos \frac{2\pi}{n}$: $\cos \frac{2\pi}{n} \in \mathbb{Q}$ forces $n \in \{3,4,6\}$. Hexagon: permutations of $(1,-1,0)$. Plane: area of triangle/hexagon has a $\sqrt{3}$.
7. $\binom{i+j}{i}=\sum_k \binom{i}{k}\binom{j}{k}$: $M=LL^T$ with L unitriangular.
8. On the sphere of radius 1 above the disc, a strip of width w lifts to a zone of area $2\pi w$ (Archimedes). Covering the hemisphere-pair requires $\sum 2\pi w_i \geq 4\pi$.
9. Each $(a_i-m)(a_i-M) \leq 0$ gives $a_i+\frac{mM}{a_i} \leq m+M$; AM-GM on $\sum a_i$ and $mM\sum \frac{1}{a_i}$. Equality: half at m, half at M (needs n even).
10. Dominated convergence on $\sum_j (1-\frac{j}{n})^n$. $C=\frac{e(e+1)}{2(e-1)^3}$ from $(1-\frac{j}{n})^n=e^{-j}(1-\frac{j^2}{2n}+\dots)$.

Mock 22 · CHI

1. Insert vertices one at a time into a Hamiltonian path. Camion: extend a cycle of length $k<n$ to $k+1$ using strong connectivity.
2. $\frac{1}{m+n}=\int_0^1 t^{m+n-1}dt$ and Schur's test with weights $m^{-1/2}$; sharpness with $a_n=n^{-1/2-\varepsilon}$.
3. $\frac{\pi^2}{12}-\frac{\ln^2 2}{2}$. Use $\sum H_n x^n=\frac{-\ln(1-x)}{1-x}$ and integrate.
4. Erdős: count $n \leq N$ with all prime factors among the first k primes, at most $2^k\sqrt{N}$; if the tail sum were $<\frac{1}{2}$, the rest are at most $\frac{N}{2}$.
5. Linearity of expectation: E is additive over pieces, so $E=cL$; a circle of diameter d gives $c=\frac{2}{\pi d}$. A constant-width curve crosses exactly twice when $w=d$.
6. Points on a circle at angles $2\theta_k$ with rational $\cos, \sin$ of each θ_k (Pythagorean angles): chords $2R|\sin(\theta_j-\theta_k)|$ are rational; scale.
7. Hadamard: rows have length $\sqrt{n}$, $|\det| \leq \prod |r_i|$. Equality needs orthogonal rows; normalise three rows to get $4 \mid n$.
8. $|x_i+x_j|-|x_i-x_j|=\frac{2}{\pi}\int \frac{\sin x_i t \sin x_j t}{t^2}dt$; summing gives $\frac{2}{\pi}\int \frac{(\sum \sin x_i t)^2}{t^2} \geq 0$.
9. Random colouring; union bound. For $n=\lfloor 2^{k/2} \rfloor$: $\binom{n}{k}2^{1-\binom{k}{2}}<\frac{2^{1+k/2}}{k!}<1$.
10. Induct on m: $k \mapsto k+2^{m-1}$ flips signs, and $(k+2^{m-1})^j-k^j$ has lower degree. Split by parity of $s(k)$: $\{0,3,5,6,9,10,12,15\}$ vs $\{1,2,4,7,8,11,13,14\}$.

Mock 23 · PSI

1. Katona: for a cyclic order, at most k intervals of length k are pairwise intersecting. Double count (set, cyclic order) pairs.
2. Apply extreme value to $f(x)-cx$ on $[a,b]$. Example: $x^2 \sin \frac{1}{x}$ (0 at 0).
3. Ahmed's integral: $\frac{5\pi^2}{96}$. Write $\frac{\arctan \sqrt{2+x^2}}{\sqrt{2+x^2}}$ as an integral, swap order, use symmetry.
4. Parametrise the conic from $(1,0)$ with slope t: $p-\left(\frac{-1}{p}\right)$ points; equivalently $\sum_x (1+(\frac{1-x^2}{p}))$.
5. A k-cycle with $k>\frac{n}{2}$ is unique if it exists; expected number of k-cycles is $\frac{1}{k}$. $\sum_{n/2<k\leq n}\frac{1}{k} \to \ln 2$.
6. Radon's theorem for 4 points in the plane, then induct on m.

7. Subtract row 1 from others and factor; induction. Or: both sides are rational functions with the same poles and zeros.

8. Iterates $x_n = f^n(x)$ satisfy $x_{n+2} = 6x_n - x_{n+1}$: $x_n = A2^n + B(-3)^n$; positivity for all n forces $B = 0$, so $f(x) = 2x$.

9. Among three vectors, two lines meet at $\leq 60°$; replace them by $\pm(v_i - v_j)$, length ≤ 1. Two left: choose signs with angle $\geq 90°$. Average of $|\sum \varepsilon_i v_i|^2$ is n.

10. Hyperbola method: $\sum_{ab\leq x} 1 = 2\sum_{a\leq\sqrt{x}} \lfloor \frac{x}{a} \rfloor - \lfloor\sqrt{x}\rfloor^2$.

Mock 24 · OMEGA

1. Monsky: extend the 2-adic valuation to $\mathbb{R}$ and three-colour the plane by it. Sperner's lemma gives a rainbow triangle in the dissection, and a rainbow triangle has area of 2-adic absolute value at least 4. But $\frac{1}{n}$ with n odd has 2-adic absolute value 1.

2. Baire: $\mathbb{R} = \bigcup_n E_n$, $E_n = \{f^{(n)} = 0\}$ closed. On every interval some E_n has interior; the open set where f is locally polynomial is dense, and its complement (a closed set with no isolated points if nonempty) leads to a contradiction by applying Baire again on it.

3. Niven: if $\pi^2 = \frac{a}{b}$, $f(x) = \frac{x^n(1-x)^n}{n!}$, $F = b^n \sum_k (-1)^k \pi^{2n-2k} f^{(2k)}$; $\pi a^n \int_0^1 f(x) \sin \pi x \, dx$ is an integer in $(0, 1)$ for large n.

4. Dvir's polynomial method: if $|K| < \binom{q+n-1}{n}$, a nonzero polynomial of degree $\leq q-1$ vanishes on K; restrict to lines to kill its top homogeneous part.

5. Erdős: bound $\binom{2n}{n} \geq \frac{4^n}{2n+1}$ against its prime factorisation, which has no primes in $(\frac{2}{3}n, n]$, and small primes contribute little.

6. Taylor at $x \pm h$: $|f'(x)| \leq \frac{A}{h} + \frac{Ch}{2}$; optimise $h = \sqrt{2A/C}$. Sharpness via a smoothed piecewise-parabolic periodic function.

7. Fisher: incidence matrix M, $MM^T = \lambda J + \mathrm{diag}(|A_i| - \lambda)$ is positive definite (at most one $|A_i| = \lambda$), so rank $m \leq n$.

8. Dehn invariant $\sum \ell \otimes \theta \in \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Q}$: cube has 0, tetrahedron $6\ell \otimes \arccos \frac{1}{3} \neq 0$ since $\arccos \frac{1}{3} \notin \pi\mathbb{Q}$ (Chebyshev: $\cos n\theta$ has denominator 3^n).

9. Uniform convergence gives continuity. At any x, dyadic intervals $u_n \leq x \leq v_n$ of length 2^{-n} have difference quotients that change by ± 1 each step, so they cannot converge.

10. $p_d < 1$ iff $\sum_n P(S_{2n} = 0) < \infty$. $P(S_{2n} = 0) \asymp n^{-d/2}$ (Fourier: $\int_{[-\pi,\pi]^d} \frac{dt}{1-\phi(t)}$ finite iff $d \geq 3$).